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Aimé Lachal

Chapter 18

A RANDOM WALK MODEL RELATED TO THE CLUSTERING OF MEMBRANE RECEPTORS

Aimé LACHAL*

Université de Lyon, CNRS, INSA-Lyon, ICJ, UMR5208, F-69621, France

Postal address:
Institut National des Sciences Appliquées de Lyon
Pôle de Mathématiques/Institut Camille Jordan
Bâtiment Léonard de Vinci, 20 avenue Albert Einstein
69621 Villeurbanne Cedex, FRANCE

Abstract

In a cellular medium, the plasmic membrane is a place of interactions between the cell and its direct external environment. A classic model describes it as a fluid mosaic. The fluid phase of the membrane allows a lateral degree of freedom to its constituents: they seem to be driven by random motions along the membrane. On the other hand, experimentations bring to light inhomogeneities on the membrane; these micro-domains (the so-called rafts) are very rich in proteins and phospholipids. Nevertheless, few functional properties of these micro-domains have been shown and it appears necessary to build appropriate models of the membrane for recreating the biological mechanism.

In this article, we propose a random walk model simulating the evolution of certain constituents–the so-called ligands–along a heterogeneous membrane. Inhomogeneities–the rafts–are described as being still clustered receptors. An important variable of interest to biologists is the time that ligands and receptors bind during a fixed amount of time. This stochastic time can be interpreted as a measurement of affinity/sentivity of ligands for receptors. It corresponds to the sojourn time in a suitable set for a certain random walk.

We provide a method of calculation for the probability distribution of this random variable and we next determine explicitly this distribution in the simple case when we are dealing with only one ligand and one receptor. We finally address some further more realistic models.

Key Words: random walk, ligand, receptor, sojourn time, generating function.

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*E-mail address: aime.lachal@insa-lyon.fr

1 Introduction

1.1 The biological context

In a cellular medium, the plasmic membrane is a place of interactions between the cell and its direct external environment. A classic model describes it as a fluid mosaic. The fluid phase of the membrane allows a lateral degree of freedom to its constituents: they seem to be driven by random motions along the membrane. As a first estimate, the membrane could be viewed as a two-dimensional manifold on which the constituents are driven by a Brownian motion. This makes the local concentration of constituents independent from their localization on the membrane. In other words, the membrane should be homogeneous. In fact, experimentations brought to light inhomogeneities on the membrane. Indeed, different liquid phases were observed. Certain constituents tend to group in clusters inside membrane domains, forming dense receptor spots (sometimes called *rafts*) and depleted zones elsewhere, instead of covering homogeneously the membrane surface. These microdomains are generally very rich in proteins and phospholipids, but few functional properties have been shown. Nowadays, the systematical presence of certain proteins and lipids in different liquid phases has become a marker of such inhomogeneities.

Cellular response to changes in the concentration of different chemical species (the so-called *ligands*) in the extracellular medium is induced by the ligand binding to dedicated transmembrane receptors. The receptor-ligand binding is based on local physical interactions, ligand molecules randomly roam in the extracellular medium until they meet a receptor at the cell surface and possibly dock. The binding mechanism, an important feature for the biologists, provides a way of measurement of the affinity/sensitivity of the ligands for the receptors.

We refer the reader to the paper by Caré & Soula [2] for an accurate description of the biological context.

In this article, we simplify the cellular medium by reducing it to the dimension one and by discretizing Brownian motion, that is we propose a naive one-dimensional random walk model to simulate the extracellular traffic of ligands along a heterogeneous membrane. Inhomogeneities–the rafts–are described as being a set of still monovalent receptors on the membrane.

An important variable of interest to biologists in the aforementioned binding mechanism is the *docking-time*, this is the time that ligands and receptors dock. Actually, this variable is proportional to the time spent by ligands on receptors during a fixed amount of time–the duration of observation–with the rule that if several ligands meet simultaneously the same receptor, they are counted only one time because of the effect of the steric hindrance.

We propose a method of calculation for the probability distribution of this random variable. Next, we shall consider particular cases and we shall determine explicitly this distribution in the simple case when we are dealing with only one ligand and one receptor. Finally, we sketch more realistic models by introducing a circular random walk or Brownian motion and we give some information about how to extend the results we have obtained for the one-dimensional random walk to these cases.

1.2 The random walk model

The cellular boundary is modeled as the (discrete) integer line $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$. Set $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}^* = \mathbb{N}\backslash\{0\} = \{1, 2, \dots\}$. We are given r receptors $a_1, \dots, a_r \in \mathbb{Z}$ such that $a_1 < \cdots < a_r$, $\mathcal{R} = \{a_1, \dots, a_r\}$ and ℓ ligands which are modeled as ℓ independent identically distributed Bernoulli random walks with parameter $p \in (0, 1)$. The steps of each random walk take the value $+1$ with probability p and -1 with the probability $q = 1 - p$. The ℓ ligands thus induce an ℓ-dimensional random walk $(\mathbf{S}(\iota))_{\iota\in\mathbb{N}}$ on $\mathbb{Z}^\ell$ with, for any $\iota \in \mathbb{N}$,

$$\mathbf{S}(\iota) = (S_1(\iota), \dots, S_\ell(\iota)).$$

Set $\boldsymbol{\mathcal{E}} = \bigcup_{j=1}^{\ell} \boldsymbol{\mathcal{E}}_j$ where the $\boldsymbol{\mathcal{E}}_j$'s are defined by

$$\boldsymbol{\mathcal{E}}_j = \{(x_1, \dots, x_\ell) \in \mathbb{Z}^\ell : x_j \in \mathcal{R}\} = \mathbb{Z}^{j-1} \times \mathcal{R} \times \mathbb{Z}^{\ell-j}.$$

The $\boldsymbol{\mathcal{E}}_j$'s can be viewed as unions of hyperplanes parallel to the j-th axis of coordinates; indeed, $\boldsymbol{\mathcal{E}}_j = \bigcup_{i=1}^{r} \boldsymbol{\mathcal{E}}_{ij}$ where the $\boldsymbol{\mathcal{E}}_{ij}$'s are the hyperplanes

$$\boldsymbol{\mathcal{E}}_{ij} = \{(x_1, \dots, x_\ell) \in \mathbb{Z}^\ell : x_j = a_i\} = \mathbb{Z}^{j-1} \times \{a_i\} \times \mathbb{Z}^{\ell-j}.$$

The time spent by ligands on certain receptors during a fixed amount of time, say n ($n \in \mathbb{N}^*$), is given by

$$\mathbf{T}_{n,\boldsymbol{\mathcal{E}}} = \#\{\iota \in \{1, \dots, n\} : \mathbf{S}(\iota) \in \boldsymbol{\mathcal{E}}\} = \sum_{\iota=1}^{n} 1\!\!1_{\{\mathbf{S}(\iota)\in\boldsymbol{\mathcal{E}}\}},$$

this is the sojourn time of the random walk $(\mathbf{S}(\iota))_{\iota\in\mathbb{N}}$ in the set $\boldsymbol{\mathcal{E}}$ up to time n. Indeed, $\mathbf{S}(\iota) \in \boldsymbol{\mathcal{E}}$ means that there exists two indices $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, \ell\}$ such that $S_j(\iota) = a_i$, that is, at least one ligand is located at one receptor. Moreover, if several ligands $S_{j_1}, \dots, S_{j_k}$ meet simultaneously a same receptor a_i at a certain time ι, i.e., $S_{j_1}(\iota) = \cdots = S_{j_k}(\iota) = a_i$, then $\mathbf{S}(\iota) \in \bigcup_{j=1}^{\ell} \boldsymbol{\mathcal{E}}_{ij}$ and they are thus counted only one time in the quantity $1\!\!1_{\{\mathbf{S}(\iota)\in\boldsymbol{\mathcal{E}}\}}$.

It is clear that the computations of the expectation and the variance of $\mathbf{T}_{n,\boldsymbol{\mathcal{E}}}$ are easy. Computing the probability distribution of $\mathbf{T}_{n,\boldsymbol{\mathcal{E}}}$ is quite more complicated. Our aim is to describe a possible way for computing this latter. We provide matrix equations which may be solved by using numerical schemes.

1.3 Settings

We shall adopt the following convention for the settings: the roman letters will be related to the dimension 1 (scalars) while the bold letters will be related to the dimension ℓ (vectors or matrices). We shall also put the space variables ($x, y, z \in \mathbb{Z}, \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{Z}^\ell$, etc.) in subscript and the time variables ($\iota, \jmath, k, n$, etc.) in superscript. The letters i, j, l, m, etc.

will be used as indices, e.g., for labeling the receptors. Be aware of the difference between ι and i, between $\jmath$ and j. The letters $p, q, r, \mathbf{p}, \mathbf{q}, \mathbf{r}$, etc. will be used for defining certain probabilities, the letters $G, H, K, \mathbf{G}, \mathbf{H}, \mathbf{K}$, etc. for defining certain generating functions or matrices. the letters $S, \mathbf{S}$ for defining random walks, the letters $n, \tau, T, \mathbf{T}$ for defining certain times.

The settings $\mathbf{P}_{\mathbf{x}}$ and $\mathbf{E}_{\mathbf{x}}$ denote the probability and expectation related to the ℓ-dimensional random walk $(\mathbf{S}(\iota))_{\iota\in\mathbb{N}}$ started at a point $\mathbf{x} \in \mathbb{Z}^\ell$ at time 0. When the index of a one-dimensional random walk $(S_j(\iota))_{\iota\in\mathbb{N}}$ will not be used, we shall relabel it as any generic one-dimensional random walk: $(S(\iota))_{\iota\in\mathbb{N}}$. The settings $\mathbb{P}_x$ and $\mathbb{E}_x$ denote the probability and expectation related to the random walk $(S(\iota))_{\iota\in\mathbb{N}}$ started at a point $x \in \mathbb{Z}$ at time 0.

We introduce the first hitting time of the random walk $(\mathbf{S}(\iota))_{\iota\in\mathbb{N}}$ in $\boldsymbol{\mathcal{E}}$:

$$\boldsymbol{\tau}_{\boldsymbol{\mathcal{E}}} = \min\{\iota \in \mathbb{N}^* : \mathbf{S}(\iota) \in \boldsymbol{\mathcal{E}}\}$$

if there exists an index ι such that $\mathbf{S}(\iota) \in \boldsymbol{\mathcal{E}}$; else we set $\boldsymbol{\tau}_{\boldsymbol{\mathcal{E}}} = +\infty$. Similarly, we introduce the first hitting times of the single or multiple levels a_j, $a_{j'}$, $a_{j''}$ together with that of the set $\mathcal{R}$ for the random walk $(S(\iota))_{\iota\in\mathbb{N}}$:

$$\begin{aligned}
\tau_{a_j} &= \min\{\iota \in \mathbb{N}^* : S(\iota) = a_j\},\\
\tau_{a_j,a_{j'}} &= \min\{\iota \in \mathbb{N}^* : S(\iota) \in \{a_j, a_{j'}\}\} = \min(\tau_{a_j}, \tau_{a_{j'}}),\\
\tau_{a_j,a_{j'},a_{j''}} &= \min\{\iota \in \mathbb{N}^* : S(\iota) \in \{a_j, a_{j'}, a_{j''}\}\} = \min(\tau_{a_j}, \tau_{a_{j'}}, \tau_{a_{j''}}),\\
\tau_{\mathcal{R}} &= \min\{\iota \in \mathbb{N}^* : S(\iota) \in \mathcal{R}\} = \min(\tau_{a_1}, \dots, \tau_{a_r})
\end{aligned}$$

with the same convention: $\min(\emptyset) = +\infty$.

Let us define several family of probabilities: for $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^\ell$, $\iota, k, n \in \mathbb{N}$ and $\jmath \in \mathbb{N}^*$,

$$\begin{aligned}
\mathbf{p}_{\mathbf{x},\mathbf{y}}^{(\iota)} &= \mathbf{P}_{\mathbf{x}}\{\mathbf{S}(\iota) = \mathbf{y}\} = \mathbf{P}_{\mathbf{0}}\{\mathbf{S}(\iota) = \mathbf{y} - \mathbf{x}\},\\
\mathbf{q}_{\mathbf{x},\mathbf{y}}^{(\jmath)} &= \mathbf{P}_{\mathbf{x}}\{\boldsymbol{\tau}_{\boldsymbol{\mathcal{E}}} = \jmath, \mathbf{S}(\boldsymbol{\tau}_{\boldsymbol{\mathcal{E}}}) = \mathbf{y}\},\\
\mathbf{r}_{\mathbf{x}}^{(k,n)} &= \mathbf{P}_{\mathbf{x}}\{\mathbf{T}_{n,\boldsymbol{\mathcal{E}}} = k\},
\end{aligned}$$

We plainly have $\mathbf{q}_{\mathbf{x},\mathbf{y}}^{(\jmath)} = 0$ if $\mathbf{y} \notin \boldsymbol{\mathcal{E}}$ and $\mathbf{r}_{\mathbf{x}}^{(k,n)} = 0$ if $k > n$. We also introduce their related generating functions: for $|u|, |v| < 1$,

$$\begin{aligned}
G_{\mathbf{x},\mathbf{y}}(u) &= \sum_{\iota=0}^{\infty} \mathbf{p}_{\mathbf{x},\mathbf{y}}^{(\iota)} u^\iota,\\
H_{\mathbf{x},\mathbf{y}}(u) &= \sum_{\jmath=1}^{\infty} \mathbf{q}_{\mathbf{x},\mathbf{y}}^{(\jmath)} u^\jmath = \mathbf{E}_{\mathbf{x}}(u^{\boldsymbol{\tau}_{\boldsymbol{\mathcal{E}}}}, \mathbf{S}(\boldsymbol{\tau}_{\boldsymbol{\mathcal{E}}}) = \mathbf{y}),\\
K_{\mathbf{x}}(u, v) &= \sum_{\substack{k,n\in\mathbb{N}:\\ k\le n}} \mathbf{r}_{\mathbf{x}}^{(k,n)} u^k v^n = \sum_{n=0}^{\infty} \mathbf{E}_{\mathbf{x}}(u^{\mathbf{T}_{n,\boldsymbol{\mathcal{E}}}} v^n),
\end{aligned}$$

together with their associated matrices:

$$\mathbf{G}(u) = (G_{\mathbf{x},\mathbf{y}}(u))_{\mathbf{x},\mathbf{y}\in\boldsymbol{\mathcal{E}}}, \quad \mathbf{H}(u) = (H_{\mathbf{x},\mathbf{y}}(u))_{\mathbf{x},\mathbf{y}\in\boldsymbol{\mathcal{E}}}, \quad \mathbf{K}(u, v) = (K_{\mathbf{x}}(u, v))_{\mathbf{x}\in\boldsymbol{\mathcal{E}}}.$$

The matrices $\mathbf{G}$ and $\mathbf{H}$ are infinite squared-matrices and $\mathbf{K}$ is an infinite column-matrix.

1.4 Background on 1D-random walk

In this part, we supply several well-known results concerning classical distributions related to the one-dimensional Bernoulli random walk. We refer to [3], [5] or [6]. In order to facilitate the reading and to make the paper self-contained, we shall provide many details. In particular, we shall focus on the proofs which use generating functions since they serve as a model of the main tool we shall use throughout the paper. The proofs which do not involve any generating functions will be postponed to Appendices A and B.

1.4.1 Location of the random walk

Set, for $x, y \in \mathbb{N}$ and $\iota \in \mathbb{N}$,

$$p_{x,y}^{(\iota)} = \mathbb{P}_x\{S(\iota) = y\}, \quad G_{x,y}(u) = \sum_{\iota=0}^{\infty} p_{x,y}^{(\iota)} u^\iota.$$

We have

$$p_{x,y}^{(\iota)} = \binom{\iota}{\frac{\iota+x-y}{2}} p^{(\iota+y-x)/2} q^{(\iota+x-y)/2} = \binom{\iota}{\frac{\iota+x-y}{2}} (pq)^{\iota/2} \left(\frac{p}{q}\right)^{(y-x)/2}$$

with the convention that $\binom{\iota}{\alpha} = 0$ for $\alpha \notin \mathbb{N}$ or $\alpha > \iota$. By putting $\varpi_{x,y} = p$ if $x > y$, $\varpi_{x,y} = q$ if $x < y$, $\varpi_{x,y} = 1$ if $x = y$, we can rewrite $p_{x,y}^{(\iota)}$ as

$$p_{x,y}^{(\iota)} = \binom{\iota}{\frac{\iota+|x-y|}{2}} (pq)^{\iota/2} \left(\frac{pq}{\varpi_{x,y}^2}\right)^{|x-y|/2}$$

and next

$$G_{x,y}(u) = \left(\frac{pq}{\varpi_{x,y}^2}\right)^{|x-y|/2} \sum_{\substack{\iota\in\mathbb{N}:\, \iota\geq|x-y|,\\ \iota \text{ and } x-y \text{ with same parity}}} \binom{\iota}{\frac{\iota+|x-y|}{2}} (pq)^{\iota/2} u^\iota.$$

By performing the change of index $\iota \mapsto 2\iota + |x-y|$ in the foregoing sum, we derive

$$G_{x,y}(u) = \left(\frac{pqu}{\varpi_{x,y}}\right)^{|x-y|} \sum_{\iota=0}^{\infty} \binom{2\iota + |x-y|}{\iota} (pqu^2)^\iota. \tag{1.1}$$

In order to simplify this last sum, we shall make use of the hypergeometric function $F\left(\frac{l+1}{2}, \frac{l+2}{2}; l+1; \zeta\right)$ defined, with the usual notation $a_m = a(a+1)(a+2)\dots(a+m-1)$, by

$$F\left(\frac{l+1}{2}, \frac{l+2}{2}; l+1; \zeta\right) = \sum_{m=0}^{\infty} \frac{(\frac{l+1}{2})_m (\frac{l+2}{2})_m}{(l+1)_m m!} \zeta^m = \sum_{m=0}^{\infty} \binom{2m+l}{m} \left(\frac{\zeta}{4}\right)^m.$$

By invoking Formula 15.1.14 of [1], p. 556, namely

$$F\left(\frac{l+1}{2}, \frac{l+2}{2}; l+1; \zeta\right) = \frac{2^l}{\left(1+\sqrt{1-\zeta}\right)^l \sqrt{1-\zeta}} = \frac{2^l}{\sqrt{1-\zeta}} \left(\frac{1-\sqrt{1-\zeta}}{\zeta}\right)^l,$$

we derive the relationship

$$\sum_{m=0}^{\infty}\binom{2m+l}{m}\zeta^m=\frac{1}{\sqrt{1-4\zeta}}\left(\frac{1-\sqrt{1-4\zeta}}{2\zeta}\right)^l. \tag{1.2}$$

Set

$$A(u)=\sqrt{1-4pqu^2},\quad B^+(u)=\frac{1+A(u)}{2pu}=\frac{2qu}{1-A(u)},\quad B^-(u)=\frac{1-A(u)}{2pu}.$$

We deduce from (1.2) and (1.1) the following expression of $G_{x,y}(u)$:

$$G_{x,y}(u)=\begin{cases}\dfrac{[B^-(u)]^{x-y}}{A(u)} & \text{if } x>y,\\ \dfrac{1}{A(u)} & \text{if } x=y,\\ \dfrac{[B^+(u)]^{x-y}}{A(u)} & \text{if } x<y.\end{cases} \tag{1.3}$$

1.4.2 Hitting times of the random walk

Let us now consider the family of hitting times related to $(S(\iota))_{\iota\in\mathbb{N}}$ started at $x\in\mathbb{Z}$: for any $a,b,c\in\mathbb{Z}$ such that $a<b<c$, set

$$\begin{aligned}\tau_a&=\min\{\iota\geq 1: S(\iota)=a\},\\ \tau_{a,b}&=\min\{\iota\geq 1: S(\iota)\in\{a,b\}\},\\ \tau_{a,b,c}&=\min\{\iota\geq 1: S(\iota)\in\{a,b,c\}\}.\end{aligned}$$

We still adopt the convention $\min\emptyset=+\infty$. In certain cases depending on the starting point x, certain hitting times are related. In fact, we have

$$\tau_{a,b}=\begin{cases}\tau_a & \text{if } x<a,\\ \tau_b & \text{if } x>b,\end{cases}$$

and

$$\tau_{a,b,c}=\begin{cases}\tau_a & \text{if } x<a,\\ \tau_{a,b} & \text{if } x\in[a,b),\\ \tau_{b,c} & \text{if } x\in(b,c],\\ \tau_c & \text{if } x>c.\end{cases}$$

Notice that when the starting point is a, time τ_a is the return time to level a; when the starting point is b, time $\tau_{a,b,c}$ depends on times $\tau_{a,b}$ and $\tau_{b,c}$. Let us introduce the probabilities

$$\begin{aligned}q_{x,a}^{(\jmath)}&=\mathbb{P}_x\{\tau_a=\jmath\},\\ q_{x,a,b}^{(\jmath)-}&=\mathbb{P}_x\{\tau_{a,b}=\jmath, S(\tau_{a,b})=a\}=\mathbb{P}_x\{\tau_a=\jmath,\tau_a<\tau_b\},\\ q_{x,a,b}^{(\jmath)+}&=\mathbb{P}_x\{\tau_{a,b}=\jmath, S(\tau_{a,b})=b\}=\mathbb{P}_x\{\tau_b=\jmath,\tau_b<\tau_a\},\end{aligned}$$

together with their related generating functions:

$$H_{x,a}(u) = \sum_{\jmath=1}^{\infty} q_{x,a}^{(\jmath)} u^{\jmath} = \mathbb{E}_x(u^{\tau_a}),$$
$$H_{x,a,b}^{-}(u) = \sum_{\jmath=1}^{\infty} q_{x,a,b}^{(\jmath)-} u^{\jmath} = \mathbb{E}_x(u^{\tau_a}, \tau_a < \tau_b),$$
$$H_{x,a,b}^{+}(u) = \sum_{\jmath=1}^{\infty} q_{x,a,b}^{(\jmath)+} u^{\jmath} = \mathbb{E}_x(u^{\tau_b}, \tau_b < \tau_a).$$

One-sided threshold

For calculating the probability distribution of τ_a, we invoke a "continuity" argument by observing that, for $\jmath \in \mathbb{N}^*$, if $S(\jmath) = a$ then $\tau_a \leq \jmath$, that is there exists an index $l \in \{1, \dots, \jmath\}$ such that $\tau_a = l$. This leads to the following relationship:

$$\mathbb{P}_x\{S(\jmath) = a\} = \mathbb{P}_x\{S(\jmath) = a, \tau_a \leq \jmath\} = \sum_{l=1}^{\jmath} \mathbb{P}_x\{\tau_a = l\}\mathbb{P}_a\{S(\jmath - l) = a\}$$

or, equivalently,

$$p_{x,a}^{(\jmath)} = \sum_{l=1}^{\jmath} q_{x,a}^{(l)} p_{a,a}^{(\jmath-l)}.$$

Then, the generating functions satisfies the equation

$$G_{x,a}(u) = \delta_{a,x} + H_{x,a}(u) G_{a,a}(u)$$

from which we deduce, by (1.3),

$$H_{x,a}(u) = \begin{cases} \dfrac{G_{x,a}(u)}{G_{a,a}(u)} & \text{if } x \neq a, \\[2ex] 1 - \dfrac{1}{G_{a,a}(u)} & \text{if } x = a, \end{cases} \tag{1.4}$$

$$= \begin{cases} [B^{-}(u)]^{x-a} & \text{if } x > a, \\[1ex] 1 - A(u) & \text{if } x = a, \\[1ex] [B^{+}(u)]^{x-a} & \text{if } x < a. \end{cases} \tag{1.5}$$

By using the hypergeometric function $F\left(\frac{l}{2}, \frac{l+1}{2}; l+1; \zeta\right)$ defined by

$$F\left(\frac{l}{2}, \frac{l+1}{2}; l+1; \zeta\right) = \sum_{m=0}^{\infty} \frac{(\frac{l}{2})_m (\frac{l+1}{2})_m}{(l+1)_m m!} \zeta^m = \sum_{m=0}^{\infty} \frac{l}{2m+l} \binom{2m+l}{m} \left(\frac{\zeta}{4}\right)^m.$$

and referring to Formula 15.1.13 of [1], p. 556, namely

$$F\left(\frac{l}{2}, \frac{l+1}{2}; l+1; \zeta\right) = \frac{2^l}{\left(1 + \sqrt{1-\zeta}\right)^l} = 2^l \left(\frac{1 - \sqrt{1-\zeta}}{\zeta}\right)^l,$$

we derive the relationship

$$\sum_{m=0}^{\infty} \frac{l}{2m+l}\binom{2m+l}{m}\zeta^m = \left(\frac{1-\sqrt{1-4\zeta}}{2\zeta}\right)^l. \tag{1.6}$$

We then extract from (1.5) and (1.6), for $x \neq a$,

$$\begin{aligned} H_{x,a}(u) &= \left(\frac{pqu}{\varpi_{x,a}}\right)^{|x-a|} \sum_{\jmath=0}^{\infty} \frac{|x-a|}{2\jmath+|x-a|}\binom{2\jmath+|x-a|}{\jmath}(pqu^2)^{\jmath} \\ &= \left(\frac{pq}{\varpi_{x,a}^2}\right)^{|x-a|/2} \sum_{\substack{\jmath\in\mathbb{N}:\,\jmath\geq|x-a|,\\ \jmath \text{ and } x-a \text{ with same parity}}} \frac{|x-a|}{\jmath}\binom{\jmath}{\frac{\jmath+|x-a|}{2}}(pq)^{\jmath/2}u^{\jmath}. \end{aligned}$$

We finally obtain

$$q_{x,a}^{(\jmath)} = \frac{|x-a|}{\jmath}\binom{\jmath}{\frac{\jmath+|x-a|}{2}}(pq)^{\jmath/2}\left(\frac{pq}{\varpi_{x,a}^2}\right)^{|x-a|/2}$$

that is, for $x \neq a$,

$$q_{x,a}^{(\jmath)} = \frac{|x-a|}{\jmath}\,p_{x,a}^{(\jmath)} = \frac{|x-a|}{\jmath}\binom{\jmath}{\frac{\jmath+x-a}{2}}p^{(\jmath+a-x)/2}q^{(\jmath+x-a)/2}.$$

For $x = a$, we have, by considering the location of the first step of the walk,

$$\mathbb{P}_a\{\tau_a = \jmath\} = p\,\mathbb{P}_{a+1}\{\tau_a = \jmath-1\} + q\,\mathbb{P}_{a-1}\{\tau_a = \jmath-1\}$$

which leads to

$$q_{a,a}^{(\jmath)} = \frac{1}{\jmath}\binom{\jmath}{\frac{\jmath+1}{2}}(pq)^{(\jmath+1)/2}.$$

Two-sided threshold

For calculating the probability distribution of $\tau_{a,b}$, we use the strong Markov property by writing that for $x \in (a,b)$

$$\begin{aligned} H_{x,a}(u) &= \mathbb{E}_x(u^{\tau_a}) = \mathbb{E}_x(u^{\tau_a}, \tau_a < \tau_b) + \mathbb{E}_x(u^{\tau_a}, \tau_b < \tau_a) \\ &= \mathbb{E}_x(u^{\tau_a}, \tau_a < \tau_b) + \mathbb{E}_x(u^{\tau_b}, \tau_b < \tau_a)\,\mathbb{E}_b(u^{\tau_a}) \\ &= H_{x,a,b}^-(u) + H_{b,a}(u)\,H_{x,a,b}^+(u). \end{aligned}$$

Similarly,

$$H_{x,b}(u) = H_{x,a,b}^+(u) + H_{a,b}(u)\,H_{x,a,b}^-(u)$$

and we obtain that $H_{x,a,b}^+(u)$ and $H_{x,a,b}^-(u)$ solves the linear system

$$\begin{cases} H_{b,a}(u)\,H_{x,a,b}^+(u) + H_{x,a,b}^-(u) = H_{x,a}(u), \\ H_{x,a,b}^+(u) + H_{a,b}(u)\,H_{x,a,b}^-(u) = H_{x,b}(u), \end{cases}$$

the solution of which writes

$$\begin{cases} H^+_{x,a,b}(u) = \dfrac{H_{a,b}(u)\,H_{x,a}(u) - H_{x,b}(u)}{H_{a,b}(u)\,H_{b,a}(u) - 1}, \\[2ex] H^-_{x,a,b}(u) = \dfrac{H_{b,a}(u)\,H_{x,b}(u) - H_{x,a}(u)}{H_{a,b}(u)\,H_{b,a}(u) - 1}. \end{cases} \tag{1.7}$$

Plugging the following expressions of $H_{x,a}(u)$ and $H_{x,b}(u)$

$$\begin{cases} H_{x,a}(u) = B^-(u)^{x-a} & \text{if } x > a \\ H_{x,b}(u) = B^+(u)^{x-b} & \text{if } x < b \end{cases}$$

into (1.7), we get that the generating function of $\tau_{a,b}$ is given by

$$\mathbb{E}_x(u^{\tau_{a,b}}) = \mathbb{E}_x(u^{\tau_a}, \tau_a < \tau_b) + \mathbb{E}_x(u^{\tau_b}, \tau_b < \tau_a)$$

where, for $x \in (a, b)$,

$$\begin{cases} \mathbb{E}_x(u^{\tau_a}, \tau_a < \tau_b) = \dfrac{B^+(u)^{x-b} - B^-(u)^{x-b}}{B^+(u)^{a-b} - B^-(u)^{a-b}} \\[2ex] \mathbb{E}_x(u^{\tau_b}, \tau_b < \tau_a) = \dfrac{B^+(u)^{x-a} - B^-(u)^{x-a}}{B^+(u)^{b-a} - B^-(u)^{b-a}} \end{cases} \tag{1.8}$$

and then

$$\mathbb{E}_x(u^{\tau_{a,b}}) = \frac{(1 - B^-(u)^{b-a})B^+(u)^{x-a} - (1 - B^+(u)^{b-a})B^-(u)^{x-a}}{B^+(u)^{b-a} - B^-(u)^{b-a}}. \tag{1.9}$$

The generating functions (1.8) can be inverted in order to write out explicitly the distribution of $\tau_{a,b}$. The inversion hinges on the decomposition of a certain rational fraction into partial fractions. The details are postponed to Appendix A. The result writes

$$\begin{aligned} q^{(\jmath)-}_{x,a,b} &= 2\left(\frac{q}{p}\right)^{(x-a)/2} \Bigg[\sum_{\substack{l\in\mathbb{N}:\\ 1\le l<(b-a)/2}} \cos^{x-a-1}\left(\frac{l\pi}{b-a}\right) \sin\left(\frac{l\pi}{b-a}\right) \\ &\qquad \sin\left(\frac{l(x-a)\pi}{b-a}\right) \left(2\sqrt{pq}\cos\left(\frac{l\pi}{b-a}\right)\right)^{\jmath} \Bigg] u^{\jmath}, \\ q^{(\jmath)+}_{x,a,b} &= 2\left(\frac{p}{q}\right)^{(x-b)/2} \Bigg[\sum_{\substack{l\in\mathbb{N}:\\ 1\le l<(b-a)/2}} \cos^{b-x-1}\left(\frac{l\pi}{b-a}\right) \sin\left(\frac{l\pi}{b-a}\right) \\ &\qquad \sin\left(\frac{l(b-x)\pi}{b-a}\right) \left(2\sqrt{pq}\cos\left(\frac{l\pi}{b-a}\right)\right)^{\jmath} \Bigg] u^{\jmath}. \end{aligned} \tag{1.10}$$

In the particular case $x = a$, we have

$$\begin{cases} \mathbb{E}_a(u^{\tau_a}, \tau_a < \tau_b) = pu\,\mathbb{E}_{a+1}(u^{\tau_a}, \tau_a < \tau_b) + qu\,\mathbb{E}_{a-1}(u^{\tau_a}, \tau_a < \tau_b) \\ \mathbb{E}_a(u^{\tau_b}, \tau_b < \tau_a) = pu\,\mathbb{E}_{a+1}(u^{\tau_b}, \tau_b < \tau_a) + qu\,\mathbb{E}_{a-1}(u^{\tau_b}, \tau_b < \tau_a) \end{cases}$$

which simplifies into

$$\begin{cases} \mathbb{E}_a(u^{\tau_a}, \tau_a < \tau_b) = pu\,\mathbb{E}_{a+1}(u^{\tau_a}, \tau_a < \tau_b) + qu\,\mathbb{E}_0(u^{\tau_1}), \\ \mathbb{E}_a(u^{\tau_b}, \tau_b < \tau_a) = pu\,\mathbb{E}_{a+1}(u^{\tau_b}, \tau_b < \tau_a). \end{cases} \tag{1.11}$$

Similarly, for $x = b$,

$$\begin{cases} \mathbb{E}_b(u^{\tau_a}, \tau_a < \tau_b) = qu\,\mathbb{E}_{b-1}(u^{\tau_a}, \tau_a < \tau_b), \\ \mathbb{E}_b(u^{\tau_b}, \tau_b < \tau_a) = qu\,\mathbb{E}_{b-1}(u^{\tau_b}, \tau_b < \tau_a) + pu\,\mathbb{E}_0(u^{\tau_{-1}}). \end{cases} \tag{1.12}$$

Three-sided threshold

At last, the probability distribution of $\tau_{a,b,c}$ is characterized by its generating function

$$\begin{aligned} \mathbb{E}_x(u^{\tau_{a,b,c}}) &= \mathbb{E}_x(u^{\tau_a}, S(\tau_{a,b,c}) = a) + \mathbb{E}_x(u^{\tau_b}, S(\tau_{a,b,c}) = b) \\ &\quad + \mathbb{E}_x(u^{\tau_c}, S(\tau_{a,b,c}) = c) \\ &= \mathbb{E}_x(u^{\tau_a}, \tau_a < \tau_{b,c}) + \mathbb{E}_x(u^{\tau_b}, \tau_b < \tau_{a,c}) + \mathbb{E}_x(u^{\tau_c}, \tau_c < \tau_{a,b}). \end{aligned}$$

We already observed that all the above generating functions can be expressed by means of those of $\tau_{a,b}$ and $\tau_{b,c}$ when the starting point x differs from b. When $x = b$, we have

$$\begin{aligned} \mathbb{E}_b(u^{\tau_{a,b,c}}, S(\tau_{a,b,c}) = b) &= \mathbb{E}_b(u^{\tau_b}, \tau_b < \tau_{a,c}) \\ &= pu\,\mathbb{E}_{b+1}(u^{\tau_b}, \tau_b < \tau_c) + qu\,\mathbb{E}_{b-1}(u^{\tau_b}, \tau_b < \tau_a). \end{aligned} \tag{1.13}$$

These probabilities can be expressed by means of (1.8).

1.4.3 Stopped random walk

In this part, we consider the families of generic "stopping"–probabilities:

- $\mathbb{P}_x\{S(\jmath) = y, \jmath \le \tau_a\}$ ($a \in \mathbb{R}$) for $x, y \in (-\infty, a]$ or $x, y \in [a, +\infty)$;
- $\mathbb{P}_x\{S(\jmath) = y, \jmath \le \tau_{a,b}\}$ ($a, b \in \mathbb{R}, a < b$) for $x, y \in [a, b]$;
- $\mathbb{P}_x\{S(\jmath) = y, \jmath \le \tau_{a,b,c}\}$ ($a, b, c \in \mathbb{R}, a < b < c$) for $x = y = b$.

These probabilities represent the distributions of the random walk stopped when reaching level a, a or b, a or b or c respectively. They can be evaluated by invoking the famous reflection principle. Since this principle will not be used in our further analysis, we postpone the intricate details to Appendix B. The result in the case of the one-sided barrier is, for $x, y < a$ or $x, y > a$,

$$\begin{aligned} \mathbb{P}_x\{S(\jmath) = y, \jmath \le \tau_a\} &= p_{x,y}^{(\jmath)} - \left(\frac{p}{q}\right)^{y-a} p_{x,2a-y}^{(\jmath)} \\ &= \left[\binom{\jmath}{\frac{\jmath+x-y}{2}} - \binom{\jmath}{\frac{\jmath+x+y}{2} - a}\right] p^{(\jmath+y-x)/2} q^{(\jmath+x-y)/2}. \end{aligned} \tag{1.14}$$

In the particular cases where $x = a$ or $y = a$, we have the following facts: for $x \in \mathbb{Z}$ and $y = a$,

$$\mathbb{P}_x\{S(\jmath) = a, \jmath \le \tau_a\} = \mathbb{P}_x\{\tau_a = \jmath\},$$

and for $x = a$ and $y \in \mathbb{Z}$,

$$\begin{aligned}\mathbb{P}_a\{S(\jmath) = y, \jmath \le \tau_a\} = p\,\mathbb{P}_{a+1}\{S(\jmath-1) = y, \jmath - 1 \le \tau_a\}\\ + q\,\mathbb{P}_{a-1}\{S(\jmath-1) = y, \jmath - 1 \le \tau_a\}.\end{aligned}$$

These last probabilities can be computed with the aid of (1.14).

In the case of the two-sided barrier, the result writes, for $x, y \in (a, b)$,

$$\begin{aligned}&\mathbb{P}_x\{S(\jmath) = y, \jmath \le \tau_{a,b}\}\\ &\quad= \sum_{l=-\infty}^{\infty} \left(\frac{p}{q}\right)^{l(b-a)} \left[p^{(\jmath)}_{x,y-2l(b-a)} - \left(\frac{p}{q}\right)^{y-a} p^{(\jmath)}_{x,2a-2l(b-a)-y}\right]\\ &\quad= p^{(\jmath+y-x)/2} q^{(\jmath+x-y)/2} \sum_{l=-\infty}^{\infty} \left[\binom{\jmath}{\frac{\jmath+x-y}{2} + l(b-a)} - \binom{\jmath}{\frac{\jmath+x+y}{2} - a + l(b-a)}\right].\end{aligned} \tag{1.15}$$

The above sum actually is finite, limited to the indices l such that $\frac{y-x-\jmath}{2(b-a)} \le l \le \frac{y-x+\jmath}{2(b-a)}$ for the first binomial coefficient and such that $\frac{2a-x-y-\jmath}{2(b-a)} \le l \le \frac{2a-x-y+\jmath}{2(b-a)}$ for the second one. In the particular case where $x \in \{a, b\}$ or $y \in \{a, b\}$, we have the following results. For $x \in [a, b]$, if $y = a$,

$$\mathbb{P}_x\{S(\jmath) = a, \jmath \le \tau_{a,b}\} = \mathbb{P}_x\{\tau_a = \jmath, \tau_a < \tau_b\} = q^{(\jmath)-}_{x,a,b},$$

and, if $y = b$,

$$\mathbb{P}_x\{S(\jmath) = b, \jmath \le \tau_{a,b}\} = q^{(\jmath)+}_{x,a,b},$$

These probabilities are given by (1.14). For $y \in [a, b]$, if $x = a$,

$$\begin{aligned}\mathbb{P}_a\{S(\jmath) = y, \jmath \le \tau_{a,b}\} = p\,\mathbb{P}_{a+1}\{S(\jmath-1) = y, \jmath - 1 \le \tau_{a,b}\}\\ + q\,\mathbb{P}_{a-1}\{S(\jmath-1) = y, \jmath - 1 \le \tau_a\},\end{aligned}$$

and if $x = b$,

$$\begin{aligned}\mathbb{P}_b\{S(\jmath) = y, \jmath \le \tau_{a,b}\} = p\,\mathbb{P}_{b+1}\{S(\jmath-1) = y, \jmath - 1 \le \tau_b\}\\ + q\,\mathbb{P}_{b-1}\{S(\jmath-1) = y, \jmath - 1 \le \tau_{a,b}\}.\end{aligned}$$

These probabilities are given by (1.14) and (1.15).

Finally, concerning the three-sided barrier, we simply have

$$\begin{aligned}\mathbb{P}_b\{S(\jmath) = b, \jmath \le \tau_{a,b,c}\} = p\,\mathbb{P}_{b+1}\{S(\jmath-1) = b, \jmath - 1 \le \tau_{b,c}\}\\ + q\,\mathbb{P}_{b+1}\{S(\jmath-1) = b, \jmath - 1 \le \tau_{a,b}\}.\end{aligned}$$

2 Methodology

The aim of this part is to describe a method of calculation for the probability distribution of $\mathbf{T}_{n,\boldsymbol{\varepsilon}}$ which could be numerically exploited.

2.1 The probability distribution of $\mathrm{S}(\iota)$

We have, for $\mathbf{x}=(x_1,\dots,x_\ell),\mathbf{y}=(y_1,\dots,y_\ell)\in\mathbb{Z}^\ell$,

$$\begin{aligned}\mathbf{p}^{(\iota)}_{\mathbf{x},\mathbf{y}}&=\mathbf{P}_{\mathbf{x}}\{\forall j\in\{1,\dots,\ell\},\,S_j(\iota)=y_j\}=\prod_{j=1}^{\ell}\mathbb{P}_{x_j}\{S(\iota)=y_j\}\\&=\left[\prod_{j=1}^{\ell}\binom{\iota}{\frac{\iota+x_j-y_j}{2}}\right]p^{\left[\iota+\sum_{j=1}^{\ell}(y_j-x_j)\right]/2}q^{\left[\iota+\sum_{j=1}^{\ell}(x_j-y_j)\right]/2}.\end{aligned}$$

In the above formula, $\mathbf{p}^{(\iota)}_{\mathbf{x},\mathbf{y}}$ does not vanish if and only if for all $j\in\{1,\dots,\ell\}$, $x_j-y_j+\iota$ is even and $|x_j-y_j|\le\iota$. Then the associated generating function is given by

$$G_{\mathbf{x},\mathbf{y}}(u)=\left(\frac{p}{q}\right)^{\left[\sum_{j=1}^{\ell}(y_j-x_j)\right]/2}\sum_{\iota=0}^{\infty}\left[\prod_{j=1}^{\ell}\binom{\iota}{\frac{\iota+x_j-y_j}{2}}\right]\left(pqu^2\right)^{\iota/2}. \tag{2.1}$$

Set

$$A(\xi_1,\dots,\xi_\ell;z)=\sum_{\substack{\iota\in\mathbb{N}:\,\iota\ge\max(|\xi_1|,\dots,|\xi_\ell|),\\ \iota,\xi_1,\dots,\xi_\ell\text{ with same parity}}}\left[\prod_{j=1}^{\ell}\binom{\iota}{\frac{\iota+\xi_j}{2}}\right]z^{\iota/2}.$$

The function A does not vanish if and only if $\xi_1,\dots,\xi_\ell$ have the same parity. By performing the change of index $\iota\mapsto 2\iota+|\xi_{j_0}|$, where j_0 is an index such that $|\xi_{j_0}|$ is the maximum of the $|\xi_1|,\dots,|\xi_\ell|$, in the sum defining A, we get

$$A(\xi_1,\dots,\xi_\ell;z)=z^{|\xi_{j_0}|}\sum_{\iota=0}^{\infty}\left[\prod_{j=1}^{\ell}\binom{2\iota+|\xi_{j_0}|}{\iota+\frac{|\xi_{j_0}|+\xi_j}{2}}\right]z^{\iota}.$$

The quantity (2.1) can be rewritten as follows.

Proposition 2.1 *For any* $\mathbf{x}=(x_1,\dots,x_\ell),\mathbf{y}=(y_1,\dots,y_\ell)\in\mathbb{Z}^\ell$,

$$G_{\mathbf{x},\mathbf{y}}(u)=\left(\frac{p}{q}\right)^{\left[\sum_{j=1}^{\ell}(y_j-x_j)\right]/2}A\big(x_1-y_1,\dots,x_\ell-y_\ell;pqu^2\big).$$

We can express the function A by means of hypergeometric functions. To see this, we set $\alpha=|\xi_{j_0}|$ and $\beta_j=\frac{|\xi_{j_0}|+\xi_j}{2}$ and assume, e.g., that $\xi_{j_0}\le 0$ so that $\beta_{j_0}=0$. In the case where $\xi_{j_0}\ge 0$, we would have $\alpha-\beta_{j_0}=0$. Invoking the duplication formula for the

Gamma function, we write

$$\begin{aligned}\prod_{j=1}^{\ell}\binom{2\iota+\alpha}{\iota+\beta_j} &= \prod_{j=1}^{\ell}\frac{\Gamma(2\iota+\alpha+1)}{\Gamma(\iota+\beta_j+1)\Gamma(\iota+\alpha-\beta_j+1)}\\ &= \frac{2^{(2\iota+\alpha)\ell}}{\pi^{\ell/2}}\prod_{j=1}^{\ell}\frac{\Gamma(\iota+\frac{\alpha+1}{2})\Gamma(\iota+\frac{\alpha+2}{2})}{\Gamma(\iota+\beta_j+1)\Gamma(\iota+\alpha-\beta_j+1)}\\ &= \frac{2^{(2\iota+\alpha)\ell}}{\pi^{\ell/2}}\frac{[\Gamma(\iota+\frac{\alpha+1}{2})\Gamma(\iota+\frac{\alpha+2}{2})]^{\ell}}{i!\prod_{1\le j\le\ell, j\neq j_0}\Gamma(\iota+\beta_j+1)\prod_{1\le j\le\ell}\Gamma(\iota+\alpha-\beta_j+1)}.\end{aligned}$$

Therefore, using the generalized hypergeometric function

$$\begin{aligned}{}_sF_t\left(\begin{matrix}\alpha_1,\ldots,\alpha_s\\ \beta_1,\ldots,\beta_t\end{matrix};z\right) &= \sum_{m=0}^{\infty}\frac{(\alpha_1)_m\ldots(\alpha_s)_m}{(\beta_1)_m\ldots(\beta_t)_m}\frac{z^m}{m!}\\ &= \frac{\Gamma(\beta_1)\ldots\Gamma(\beta_t)}{\Gamma(\alpha_1)\ldots\Gamma(\alpha_s)}\sum_{m=0}^{\infty}\frac{\Gamma(m+\alpha_1)\ldots\Gamma(m+\alpha_s)}{\Gamma(m+\beta_1)\ldots\Gamma(m+\beta_t)}\frac{z^m}{m!},\end{aligned}$$

we obtain

$$\begin{aligned}&A(\xi_1,\ldots,\xi_\ell;z)\\ &\quad= \frac{2^{\alpha\ell}}{\pi^{\ell/2}}\frac{[\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{\alpha+2}{2})]^{\ell}}{\prod_{1\le j\le\ell, j\neq j_0}\Gamma(\beta_j+1)\prod_{1\le j\le\ell}\Gamma(\alpha-\beta_j+1)}z^{\alpha}\\ &\quad\times {}_{2\ell}F_{2\ell-1}\left(\begin{matrix}(\alpha+1)/2,\ldots,(\alpha+1)/2,(\alpha+2)/2,\ldots,(\alpha+2)/2\\ \beta_1+1,\ldots,\beta_\ell+1,\alpha-\beta_1+1,\ldots,\alpha-\beta_\ell+1\end{matrix};4^{\ell}z\right)\end{aligned}$$

with the convention that in the list $(\alpha+1)/2,\ldots,(\alpha+1)/2,(\alpha+2)/2,\ldots,(\alpha+2)/2$ lying within the function ${}_{2\ell}F_{2\ell-1}$ above,the terms $(\alpha+1)/2$ and $(\alpha+2)/2$ are repeated ℓ times and in the list $\beta_1+1,\ldots,\beta_\ell+1$ the term $\beta_{j_0}+1$ which equals one is evicted. Observing that the coefficient lying before the hypergeometric function can be simplified into

$$\begin{aligned}&\frac{2^{\alpha\ell}}{\pi^{\ell/2}}\frac{[\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{\alpha+2}{2})]^{\ell}}{\prod_{1\le j\le\ell, j\neq j_0}\Gamma(\beta_j+1)\prod_{1\le j\le\ell}\Gamma(\alpha-\beta_j+1)}\\ &\qquad\qquad= \frac{\alpha^{\ell}}{\prod_{j=1}^{\ell}\beta_j!\prod_{j=1}^{\ell}(\alpha-\beta_j)!} = \prod_{j=1}^{\ell}\binom{\alpha}{\beta_j},\end{aligned}$$

we finally derive the following expression of $A(\xi_1,\ldots,\xi_\ell;z)$.

Proposition 2.2 *We have*

$$\begin{aligned}&A(\xi_1,\ldots,\xi_\ell;z) = \left(\prod_{j=1}^{\ell}\binom{|\xi_{j_0}|}{\beta_j}\right)z^{|\xi_{j_0}|}\\ &\quad\times {}_{2\ell}F_{2\ell-1}\left(\begin{matrix}\frac{|\xi_{j_0}|+1}{2},\ldots,\frac{|\xi_{j_0}|+1}{2},\frac{|\xi_{j_0}|+2}{2},\ldots,\frac{|\xi_{j_0}|+2}{2}\\ \frac{|\xi_{j_0}|+\xi_1}{2}+1,\ldots,\frac{|\xi_{j_0}|+\xi_\ell}{2}+1,\frac{|\xi_{j_0}|-\xi_1}{2}+1,\ldots,\frac{|\xi_{j_0}|-\xi_\ell}{2}+1\end{matrix};4^{\ell}z\right).\end{aligned}$$

In the list $\frac{|\xi_{j_0}|+\xi_1}{2}+1,\dots,\frac{|\xi_{j_0}|+\xi_\ell}{2}+1,\frac{|\xi_{j_0}|-\xi_1}{2}+1,\dots,\frac{|\xi_{j_0}|-\xi_\ell}{2}+1$ *lying within the function* ${}_{2\ell}F_{2\ell-1}$ *above, that of the two term* $\frac{|\xi_{j_0}|+\xi_{j_0}}{2}+1$ *and* $\frac{|\xi_{j_0}|-\xi_{j_0}}{2}+1$ *which equals one is evicted.*

2.2 The probability distribution of $\boldsymbol{\tau_{\mathcal{E}}}$

Fix $\jmath\in\mathbb{N}^*$. Notice that for $\mathbf{y}=(y_1,\dots,y_\ell)\in\boldsymbol{\mathcal{E}}$, the event $\{\boldsymbol{\tau_{\mathcal{E}}}=\jmath,\mathbf{S}(\boldsymbol{\tau_{\mathcal{E}}})=\mathbf{y}\}$ means that $\mathbf{S}(\iota)\notin\boldsymbol{\mathcal{E}}$ for all $\iota\in\{0,1,\dots,\jmath-1\}$, and $\mathbf{S}(\jmath)=\mathbf{y}$. Moreover, the event $\{\mathbf{S}(\iota)\notin\boldsymbol{\mathcal{E}}\}$ is equal to $\bigcap_{j=1}^{\ell}\{S_j(\iota)\notin\mathcal{R}\}$ which means that $S_j(\iota)\notin\mathcal{R}$ for all $j\in\{1,\dots,\ell\}$. Thus

$$\begin{aligned}
\mathbf{q}_{\mathbf{x},\mathbf{y}}^{(\jmath)}&=\mathbf{P}_{\mathbf{x}}\{\forall\iota\in\{0,1,\dots,\jmath-1\},\forall j\in\{1,\dots,\ell\},\,S_j(\iota)\notin\mathcal{R}\text{ and }S_j(\jmath)=y_j\}\\
&=\prod_{j=1}^{\ell}\mathbb{P}_{x_j}\{\forall\iota\in\{0,1,\dots,\jmath-1\},\,S_j(\iota)\notin\mathcal{R}\text{ and }S_j(\jmath)=y_j\}\\
&=\prod_{j=1}^{\ell}\mathbb{P}_{x_j}\{S(\jmath)=y_j,\jmath\le\tau_{\mathcal{R}}\}.
\end{aligned}$$

The quantity $\mathbb{P}_{x_j}\{S(\jmath)=y_j,\jmath\le\tau_{\mathcal{R}}\}$ is nothing but the probability of the one-dimensional random walk stopped when reaching the set $\mathcal{R}$. Notice that one of the y_j, $1\le j\le\ell$, at least, lies in $\mathcal{R}$. It can be depicted more precisely as follows. Since the steps of the random walk are ±1, the random variable $\tau_{\mathcal{R}}$ under the probability $\mathbb{P}_{x_j}$ is the first hitting time of the nearest neighbors of x_j lying in $\mathcal{R}$, that is,

- if $x_j\in(-\infty,a_1)$ (resp. $x_j\in(a_r,+\infty)$), then $\tau_{\mathcal{R}}=\tau_{a_1}$ (resp. $\tau_{\mathcal{R}}=\tau_{a_r}$);
- if $x_j\in(a_i,a_{i+1})$ for a certain index $i\in\{2,\dots,r-1\}$, then $\tau_{\mathcal{R}}=\tau_{a_i,a_{i+1}}$;
- if $x_j=a_1$ (resp. $x_j=a_r$), then $\tau_{\mathcal{R}}=\tau_{a_1,a_2}$ (resp. τ_{a_{r-1},a_r});
- if $x_j=a_i$ for a certain index $i\in\{2,\dots,r-1\}$, then $\tau_{\mathcal{R}}=\tau_{a_{i-1},a_i,a_{i+1}}$.

As a byproduct, we have

- if $x_j\in(-\infty,a_1)$ and $y_j\in(-\infty,a_1]$ (resp. $x_j\in(a_r,+\infty)$ and $y_j\in[a_r,+\infty)$), then
$$\begin{gathered}\mathbb{P}_{x_j}\{S(\jmath)=y_j,\jmath\le\tau_{\mathcal{R}}\}=\mathbb{P}_{x_j}\{S(\jmath)=y_j,\jmath\le\tau_{a_1}\}\\ \text{(resp. }\mathbb{P}_{x_j}\{S(\jmath)=y_j,\jmath\le\tau_{\mathcal{R}}\}=\mathbb{P}_{x_j}\{S(\jmath)=y_j,\jmath\le\tau_{a_r}\});\end{gathered}$$
- if $x_j\in(a_i,a_{i+1})$ and $y_j\in[a_i,a_{i+1}]$ for a certain index $i\in\{1,\dots,r\}$,
$$\mathbb{P}_{x_j}\{S(\jmath)=y_j,\jmath\le\tau_{\mathcal{R}}\}=\mathbb{P}_{x_j}\{S(\jmath)=y_j,\jmath\le\tau_{a_i,a_{i+1}}\};$$
- if $x_j=a_1$ and $y_j\in(-\infty,a_1)$ (resp. $x_j=a_r$ and $y_j\in(a_r,+\infty)$), then
$$\begin{gathered}\mathbb{P}_{x_j}\{S(\jmath)=y_j,\jmath\le\tau_{\mathcal{R}}\}=\mathbb{P}_{x_j}\{S(\jmath)=y_j,\jmath\le\tau_{a_1}\}\\ \text{(resp. }\mathbb{P}_{x_j}\{S(\jmath)=y_j,\jmath\le\tau_{\mathcal{R}}\}=\mathbb{P}_{x_j}\{S(\jmath)=y_j,\jmath\le\tau_{a_r}\});\end{gathered}$$

- if $x_j = a_1$ and $y_j \in [a_1, a_2]$ (resp. $x_j = a_r$ and $y_j \in [a_{r-1}, a_r]$), then

$$\mathbb{P}_{x_j}\{S(\jmath) = y_j, \jmath \le \tau_{\mathcal{R}}\} = \mathbb{P}_{x_j}\{S(\jmath) = y_j, \jmath \le \tau_{a_1,a_2}\}$$
$$\text{(resp. } \mathbb{P}_{x_j}\{S(\jmath) = y_j, \jmath \le \tau_{\mathcal{R}}\} = \mathbb{P}_{x_j}\{S(\jmath) = y_j, \jmath \le \tau_{a_{r-1},a_r}\});$$

- if $x_j = a_i$ for a certain index $i \in \{2, \dots, r-1\}$ and $y \in [a_{i-1}, a_i)$ (resp. $y \in (a_i, a_{i+1}]$), then

$$\mathbb{P}_{x_j}\{S(\jmath) = y_j, \jmath \le \tau_{\mathcal{R}}\} = \mathbb{P}_{x_j}\{S(\jmath) = y_j, \jmath \le \tau_{a_{i-1},a_i}\}$$
$$\text{(resp. } \mathbb{P}_{x_j}\{S(\jmath) = y_j, \jmath \le \tau_{\mathcal{R}}\} = \mathbb{P}_{x_j}\{S(\jmath) = y_j, \jmath \le \tau_{a_i,a_{i+1}}\});$$

- if $x_j = y_j = a_i$ for a certain index $i \in \{2, \dots, r-1\}$, then

$$\mathbb{P}_{x_j}\{S(\jmath) = y_j, \jmath \le \tau_{\mathcal{R}}\} = \mathbb{P}_{x_j}\{S(\jmath) = y_j, \jmath \le \tau_{a_{i-1},a_i,a_{i+1}}\}.$$

In the other cases, the probability $\mathbb{P}_{x_j}\{S(\jmath) = y_j, \jmath \le \tau_{\mathcal{R}}\}$ vanishes.

All these probabilities belong to the following families of generic "stopping"-probabilities which are explicitly given in Section 1.4.3. With all this at hand, we can completely determine the joint probability distribution of $(\boldsymbol{\tau}_{\boldsymbol{\mathcal{E}}}, \mathbf{S}(\boldsymbol{\tau}_{\boldsymbol{\mathcal{E}}}))$.

The marginal probability distribution of $\boldsymbol{\tau}_{\boldsymbol{\mathcal{E}}}$ can be easily related to that of $\tau_{\mathcal{R}}$ according as

$$\begin{aligned}
\mathbf{P}_{\mathbf{x}}\{\boldsymbol{\tau}_{\boldsymbol{\mathcal{E}}} \ge \jmath\} &= \mathbf{P}_{\mathbf{x}}\{\forall \iota \in \{0, 1, \dots, \jmath - 1\}, \forall j \in \{1, \dots, \ell\},\, S_j(\iota) \notin \mathcal{R}\} \\
&= \prod_{j=1}^{\ell} \mathbb{P}_{x_j}\{\forall \iota \in \{0, 1, \dots, \jmath - 1\},\, S_j(\iota) \notin \mathcal{R}\} \\
&= \prod_{j=1}^{\ell} \mathbb{P}_{x_j}\{\tau_{\mathcal{R}} \ge \jmath\}.
\end{aligned}$$

The probabilities $\mathbb{P}_{x_j}\{\tau_{\mathcal{R}} \ge \jmath\}$, $1 \le j \le \ell$ can be easily computed with the aid of the distributions described in Section 1.4.2.

We propose another possible way for deriving the distribution of $(\boldsymbol{\tau}_{\boldsymbol{\mathcal{E}}}, \mathbf{S}(\boldsymbol{\tau}_{\boldsymbol{\mathcal{E}}}))$ which is characterized by the generating matrix $\mathbf{H}(u)$. If $\mathbf{S}(\iota) \in \boldsymbol{\mathcal{E}}$, then $\boldsymbol{\tau}_{\boldsymbol{\mathcal{E}}} \le \iota$. Hence, using the strong Markov property, we derive, for $\mathbf{x} \in \mathbb{Z}^\ell$ and $\mathbf{y} \in \boldsymbol{\mathcal{E}}$, the relationship, for $\iota \in \mathbb{N}^*$,

$$\mathbf{P}_{\mathbf{x}}\{\mathbf{S}(\iota) = \mathbf{y}\} = \sum_{\jmath=1}^{\iota} \sum_{\mathbf{z} \in \boldsymbol{\mathcal{E}}} \mathbf{P}_{\mathbf{x}}\{\boldsymbol{\tau}_{\boldsymbol{\mathcal{E}}} = \jmath, \mathbf{S}(\boldsymbol{\tau}_{\boldsymbol{\mathcal{E}}}) = \mathbf{z}\} \mathbf{P}_{\mathbf{z}}\{\mathbf{S}(\iota - \jmath) = \mathbf{y}\}$$

or, equivalently,

$$\mathbf{p}_{\mathbf{x},\mathbf{y}}^{(\iota)} = \sum_{\jmath=1}^{\iota} \sum_{\mathbf{z} \in \boldsymbol{\mathcal{E}}} \mathbf{q}_{\mathbf{x},\mathbf{z}}^{(\jmath)} \mathbf{p}_{\mathbf{z},\mathbf{y}}^{(\iota-\jmath)}.$$

Therefore, taking the generating functions, for $\mathbf{x} \in \mathbb{Z}^\ell$ and $\mathbf{y} \in \boldsymbol{\mathcal{E}}$,

$$G_{\mathbf{x},\mathbf{y}}(u) = \delta_{\mathbf{x},\mathbf{y}} + \sum_{\iota=1}^{\infty} \mathbf{p}_{\mathbf{x},\mathbf{y}}^{(\iota)} u^\iota = \delta_{\mathbf{x},\mathbf{y}} + \sum_{\substack{\iota,\jmath \in \mathbb{N}^*, \mathbf{z} \in \boldsymbol{\mathcal{E}}: \\ \jmath \le \iota}} \mathbf{q}_{\mathbf{x},\mathbf{z}}^{(\jmath)} \mathbf{p}_{\mathbf{z},\mathbf{y}}^{(\iota-\jmath)} u^\iota$$

which can be rewritten as

$$G_{\mathbf{x},\mathbf{y}}(u) = \delta_{\mathbf{x},\mathbf{y}} + \sum_{\mathbf{z}\in\boldsymbol{\mathcal{E}}} H_{\mathbf{x},\mathbf{z}}(u) G_{\mathbf{z},\mathbf{y}}(u), \tag{2.2}$$

which leads, when restricting ourselves to $\mathbf{x} \in \boldsymbol{\mathcal{E}}$, to the matrix equation (2.3) below. Setting $\mathbf{I}_{\boldsymbol{\mathcal{E}}}$ for the identity matrix $(\delta_{\mathbf{x},\mathbf{y}})_{\mathbf{x},\mathbf{y}\in\boldsymbol{\mathcal{E}}}$, we have the following result.

Theorem 2.3 *The generating squared-matrix $\mathbf{H}(u)$ of the numbers $\mathbf{q}_{\mathbf{x},\mathbf{y}}^{(\jmath)}$, $\mathbf{x},\mathbf{y} \in \boldsymbol{\mathcal{E}}$, $\jmath \in \mathbb{N}$, which characterizes the joint probability distribution of $(\boldsymbol{\tau}_{\boldsymbol{\mathcal{E}}}, \mathbf{S}(\boldsymbol{\tau}_{\boldsymbol{\mathcal{E}}}))$, is a solution of the following matrix equation:*

$$[\mathbf{I}_{\boldsymbol{\mathcal{E}}} - \mathbf{H}(u)]\mathbf{G}(u) = \mathbf{I}_{\boldsymbol{\mathcal{E}}}. \tag{2.3}$$

This means that the generating functions $H_{\mathbf{x},\mathbf{y}}(u)$, $\mathbf{x},\mathbf{y} \in \boldsymbol{\mathcal{E}}$, are the solutions of a system of an infinity of equations with an infinity of unknowns which seems difficult to solve.

2.3 The two first moments of $\mathbf{T}_{n,\boldsymbol{\mathcal{E}}}$

We already observed that $\{\mathbf{S}(\iota) \notin \boldsymbol{\mathcal{E}}\} = \bigcap_{j=1}^{\ell} \{S_j(\iota) \notin \mathcal{R}\}$. As a byproduct, for $\mathbf{x} = (x_1, \dots, x_\ell) \in \mathbb{Z}^\ell$,

$$\mathbf{P}_{\mathbf{x}}\{\mathbf{S}(\iota) \notin \boldsymbol{\mathcal{E}}\} = \prod_{j=1}^{\ell} \mathbb{P}_{x_j}\{S(\iota) \notin \mathcal{R}\}$$

or

$$\mathbf{P}_{\mathbf{x}}\{\mathbf{S}(\iota) \in \boldsymbol{\mathcal{E}}\} = 1 - \prod_{j=1}^{\ell}(1 - \mathbb{P}_{x_j}\{S(\iota) \in \mathcal{R}\}) = 1 - \prod_{j=1}^{\ell}\left(1 - \sum_{i=1}^{r} \mathbb{P}_{x_j}\{S(\iota) = a_i\}\right).$$

Now, the expectation of $\mathbf{T}_{n,\boldsymbol{\mathcal{E}}}$ can be easily computed as follows:

$$\mathbf{E}_{\mathbf{x}}(\mathbf{T}_{n,\boldsymbol{\mathcal{E}}}) = \sum_{\iota=1}^{n} \mathbf{P}_{\mathbf{x}}\{\mathbf{S}(\iota) \in \boldsymbol{\mathcal{E}}\} = n - \sum_{\iota=1}^{n}\prod_{j=1}^{\ell}\left(1 - \sum_{i=1}^{r} \mathbb{P}_{x_j}\{S(\iota) = a_i\}\right). \tag{2.4}$$

The second moment of $\mathbf{T}_{n,\boldsymbol{\mathcal{E}}}$ could be evaluated as follows:

$$\begin{aligned}
\mathbf{E}_{\mathbf{x}}(\mathbf{T}_{n,\boldsymbol{\mathcal{E}}}^2) &= \sum_{\iota=1}^{n}\sum_{\jmath=1}^{n} \mathbf{P}_{\mathbf{x}}\{\mathbf{S}(\iota), \mathbf{S}(\jmath) \in \boldsymbol{\mathcal{E}}\} \\
&= \sum_{\iota=1}^{n} \mathbf{P}_{\mathbf{x}}\{\mathbf{S}(\iota) \in \boldsymbol{\mathcal{E}}\} + 2 \sum_{1\le\iota<\jmath\le n} \mathbf{P}_{\mathbf{x}}\{\mathbf{S}(\iota), \mathbf{S}(\jmath) \in \boldsymbol{\mathcal{E}}\}.
\end{aligned}$$

The foregoing double sum can be computed according as

$$\begin{aligned}\sum_{1\le\iota<\jmath\le n}\mathbf{P}_{\mathbf{x}}\{\mathbf{S}(\iota),\mathbf{S}(\jmath)\in\boldsymbol{\mathcal{E}}\} &= \sum_{1\le\iota<\jmath\le n}\sum_{\mathbf{y}\in\boldsymbol{\mathcal{E}}}\mathbf{P}_{\mathbf{x}}\{\mathbf{S}(\iota)=\mathbf{y}\}\mathbf{P}_{\mathbf{y}}\{\mathbf{S}(\jmath-\iota)\in\boldsymbol{\mathcal{E}}\}\\ &= \sum_{\iota=1}^{n-1}\sum_{\mathbf{y}\in\boldsymbol{\mathcal{E}}}\mathbf{P}_{\mathbf{x}}\{\mathbf{S}(\iota)=\mathbf{y}\}\sum_{\jmath=1}^{n-\iota}\mathbf{P}_{\mathbf{y}}\{\mathbf{S}(\jmath)\in\boldsymbol{\mathcal{E}}\}\\ &= \sum_{\iota=1}^{n-1}\sum_{\mathbf{y}\in\boldsymbol{\mathcal{E}}}\mathbf{P}_{\mathbf{x}}\{\mathbf{S}(\iota)=\mathbf{y}\}\mathbf{E}_{\mathbf{y}}(\mathbf{T}_{n-\iota,\boldsymbol{\mathcal{E}}}).\end{aligned}$$

Consequently,

$$\mathbf{E}_{\mathbf{x}}(\mathbf{T}^2_{n,\boldsymbol{\mathcal{E}}})=\mathbf{E}_{\mathbf{x}}(\mathbf{T}_{n,\boldsymbol{\mathcal{E}}})+2\sum_{\iota=1}^{n-1}\sum_{\mathbf{y}\in\boldsymbol{\mathcal{E}}}\mathbf{P}_{\mathbf{x}}\{\mathbf{S}(\iota)=\mathbf{y}\}\mathbf{E}_{\mathbf{y}}(\mathbf{T}_{n-\iota,\boldsymbol{\mathcal{E}}}).$$

where $\mathbf{E}_{\mathbf{x}}(\mathbf{T}_{n,\boldsymbol{\mathcal{E}}})$ and the $\mathbf{E}_{\mathbf{y}}(\mathbf{T}_{n-\iota,\boldsymbol{\mathcal{E}}})$, $1\le\iota\le n-1$, are given by (2.4).

2.4 The probability distribution of $\mathbf{T}_{n,\boldsymbol{\mathcal{E}}}$

We now propose a way for computing the distribution of $\mathbf{T}_{n,\boldsymbol{\mathcal{E}}}$ under $\mathbf{P}_{\mathbf{x}}$ which is determined by the family of numbers $\mathbf{r}_{\mathbf{x}}^{(k,n)}$, $0\le k\le n$.

For $1\le k\le n$, if $\mathbf{T}_{n,\boldsymbol{\mathcal{E}}}=k$, then $\boldsymbol{\tau}_{\boldsymbol{\mathcal{E}}}\le n$, say $\boldsymbol{\tau}_{\boldsymbol{\mathcal{E}}}=\jmath$ for a certain $\jmath\in\{1,\dots,n\}$. Moreover, the sojourn time in $\boldsymbol{\mathcal{E}}$ up to $\boldsymbol{\tau}_{\boldsymbol{\mathcal{E}}}$ is only 1, and that after $\boldsymbol{\mathcal{E}}$ up to n, which is identical in distribution to $\mathbf{T}_{n-\jmath,\boldsymbol{\mathcal{E}}}$, equals $k-1$. Hence, using the strong Markov property, we derive the relationship, for $1\le k\le n$,

$$\mathbf{P}_{\mathbf{x}}\{\mathbf{T}_{n,\boldsymbol{\mathcal{E}}}=k\}=\sum_{\jmath=1}^{n}\sum_{\mathbf{y}\in\boldsymbol{\mathcal{E}}}\mathbf{P}_{\mathbf{x}}\{\boldsymbol{\tau}_{\boldsymbol{\mathcal{E}}}=\jmath,\mathbf{S}(\boldsymbol{\tau}_{\boldsymbol{\mathcal{E}}})=\mathbf{y}\}\mathbf{P}_{\mathbf{y}}\{\mathbf{T}_{n-\jmath,\boldsymbol{\mathcal{E}}}=k-1\}$$

or, equivalently, for $1\le k\le n$,

$$\mathbf{r}_{\mathbf{x}}^{(k,n)}=\sum_{\jmath=1}^{n}\sum_{\mathbf{y}\in\boldsymbol{\mathcal{E}}}\mathbf{q}_{\mathbf{x},\mathbf{y}}^{(\jmath)}\mathbf{r}_{\mathbf{y}}^{(k-1,n-\jmath)}.$$

Therefore, taking the generating functions, for $\mathbf{x}\in\mathbb{Z}^{\ell}$,

$$\begin{aligned}K_{\mathbf{x}}(u,v)&=\sum_{n=0}^{\infty}\mathbf{r}_{\mathbf{x}}^{(0,n)}v^n+\sum_{\substack{k,n\in\mathbb{N}^*:\\ k\le n}}\mathbf{r}_{\mathbf{x}}^{(k,n)}u^kv^n\\ &=\sum_{n=0}^{\infty}\mathbf{r}_{\mathbf{x}}^{(0,n)}v^n+\sum_{\substack{k,n,\jmath\in\mathbb{N}^*,\,\mathbf{y}\in\boldsymbol{\mathcal{E}}:\\ k\le n\text{ and }\jmath\le n+1-k}}\mathbf{q}_{\mathbf{x},\mathbf{y}}^{(\jmath)}\mathbf{r}_{\mathbf{y}}^{(k-1,n-\jmath)}u^kv^n.\end{aligned}\tag{2.5}$$

On the other hand,

$$\mathbf{r}_{\mathbf{x}}^{(0,n)}=\mathbf{P}_{\mathbf{x}}\{\mathbf{T}_{n,\boldsymbol{\mathcal{E}}}=0\}=\mathbf{P}_{\mathbf{x}}\{\boldsymbol{\tau}_{\boldsymbol{\mathcal{E}}}>n\}$$

and the corresponding generating function is

$$\begin{aligned}\sum_{n=0}^{\infty} \mathbf{r}_{\mathbf{x}}^{(0,n)} v^n &= \sum_{n=0}^{\infty} \left(1 - \sum_{k=0}^{n} \mathbf{P}_{\mathbf{x}}\{\boldsymbol{\tau}_{\boldsymbol{\mathcal{E}}} = k\}\right) v^n \\ &= \frac{1}{1-v} - \sum_{k=0}^{\infty} \mathbf{P}_{\mathbf{x}}\{\boldsymbol{\tau}_{\boldsymbol{\mathcal{E}}} = k\} \left(\sum_{n=k}^{\infty} v^n\right) \\ &= \frac{1}{1-v} - \sum_{k=0}^{\infty} \frac{v^k}{1-v} \mathbf{P}_{\mathbf{x}}\{\boldsymbol{\tau}_{\boldsymbol{\mathcal{E}}} = k\} = \frac{1 - \mathbf{E}_{\mathbf{x}}(v^{\boldsymbol{\tau}_{\boldsymbol{\mathcal{E}}}})}{1-v}. \end{aligned} \tag{2.6}$$

In view of (2.6), (2.5) can be rewritten as

$$K_{\mathbf{x}}(u,v) = \frac{1 - \mathbf{E}_{\mathbf{x}}(v^{\boldsymbol{\tau}_{\boldsymbol{\mathcal{E}}}})}{1-v} + u \sum_{\mathbf{y} \in \boldsymbol{\mathcal{E}}} H_{\mathbf{x},\mathbf{y}}(v) K_{\mathbf{y}}(u,v), \tag{2.7}$$

This leads, when restricting ourselves to $\mathbf{x} \in \boldsymbol{\mathcal{E}}$, to the following result.

Theorem 2.4 *The generating column-matrix $\mathbf{K}(u,v)$ of the family of numbers $\mathbf{P}_{\mathbf{x}}\{\mathbf{T}_{n,\boldsymbol{\mathcal{E}}} = k\}$, $\mathbf{x} \in \boldsymbol{\mathcal{E}}$, $k, n \in \mathbb{N}$, is a solution of the following matrix equation:*

$$[\mathbf{I}_{\boldsymbol{\mathcal{E}}} - u\mathbf{H}(v)]\, \mathbf{K}(u,v) = \frac{1}{1-v} [\mathbb{1}_{\boldsymbol{\mathcal{E}}} - \tilde{\mathbf{H}}(v)]. \tag{2.8}$$

The above matrix $\mathbb{1}_{\boldsymbol{\mathcal{E}}}$ is the column-matrix consisting of 1, that is $(1)_{\mathbf{x} \in \boldsymbol{\mathcal{E}}}$, $\mathbf{H}(v)$ is given by (2.3) and $\tilde{\mathbf{H}}(v)$ is defined by

$$\tilde{\mathbf{H}}(v) = \left(\mathbf{E}_{\mathbf{x}}(v^{\boldsymbol{\tau}_{\boldsymbol{\mathcal{E}}}})\right)_{\mathbf{x} \in \boldsymbol{\mathcal{E}}} = \left(\sum_{\mathbf{y} \in \boldsymbol{\mathcal{E}}} H_{\mathbf{x},\mathbf{y}}(v)\right)_{\mathbf{x} \in \boldsymbol{\mathcal{E}}} = \mathbf{H}(v) \mathbb{1}_{\boldsymbol{\mathcal{E}}}.$$

In other words, the generating matrix $K_{\mathbf{x}}(u,v)$, $\mathbf{x} \in \boldsymbol{\mathcal{E}}$, are the solutions of a system of an infinity of equations with an infinity of unknowns which seems difficult to solve.

Remark 2.5 A slightly simpler equation may be obtained by setting

$$\tilde{\mathbf{K}}(u,v) = \mathbf{K}(u,v) - \frac{1}{u(1-v)} \mathbb{1}_{\boldsymbol{\mathcal{E}}}.$$

In fact, by (2.8),

$$\begin{aligned}[\mathbf{I}_{\boldsymbol{\mathcal{E}}} - u\mathbf{H}(v)] \tilde{\mathbf{K}}(u,v) &= [\mathbf{I}_{\boldsymbol{\mathcal{E}}} - u\mathbf{H}(v)] \left[\mathbf{K}(u,v) - \frac{1}{u(1-v)} \mathbb{1}_{\boldsymbol{\mathcal{E}}}\right] \\ &= \frac{1}{1-v} [\mathbb{1}_{\boldsymbol{\mathcal{E}}} - \tilde{\mathbf{H}}(v)] - \frac{1}{u(1-v)} [\mathbb{1}_{\boldsymbol{\mathcal{E}}} - u\tilde{\mathbf{H}}(v)]. \end{aligned}$$

Thus, the modified generating function $\tilde{\mathbf{K}}(u,v)$ satisfies the matrix equation

$$[\mathbf{I}_{\boldsymbol{\mathcal{E}}} - u\mathbf{H}(v)] \tilde{\mathbf{K}}(u,v) = \frac{u-1}{u(1-v)} \mathbb{1}_{\boldsymbol{\mathcal{E}}}. \tag{2.9}$$

■

Finally, for a starting point $\mathbf{x} \in \mathbb{Z}^\ell \setminus \boldsymbol{\mathcal{E}}$, $K_{\mathbf{x}}(u,v)$ can be expressed by means of the $G_{\mathbf{x},\mathbf{y}}(u)$ and $K_{\mathbf{y}}(u,v)$, $\mathbf{y} \in \boldsymbol{\mathcal{E}}$. Indeed, by (2.2), we have for $\mathbf{x} \in \mathbb{Z}^\ell \setminus \boldsymbol{\mathcal{E}}$ and $\mathbf{y} \in \boldsymbol{\mathcal{E}}$,

$$G_{\mathbf{x},\mathbf{y}}(u) = \sum_{\mathbf{z} \in \boldsymbol{\mathcal{E}}} H_{\mathbf{x},\mathbf{z}}(u) G_{\mathbf{z},\mathbf{y}}(u),$$

that is, by introducing the row-matrices $\mathbf{G}_{\mathbf{x}}(u) = (G_{\mathbf{x},\mathbf{y}}(u))_{\mathbf{y}\in\boldsymbol{\mathcal{E}}}$ and $\mathbf{H}_{\mathbf{x}}(u) = (H_{\mathbf{x},\mathbf{y}}(u))_{\mathbf{y}\in\boldsymbol{\mathcal{E}}}$, the matrix $\mathbf{H}_{\mathbf{x}}(u)$ solves the equation

$$\mathbf{G}_{\mathbf{x}}(u) = \mathbf{H}_{\mathbf{x}}(u)\mathbf{G}(u).$$

Next (2.7) yields for $\mathbf{x} \in \mathbb{Z}^\ell \setminus \boldsymbol{\mathcal{E}}$

$$K_{\mathbf{x}}(u,v) = \frac{1 - \mathbf{E}_{\mathbf{x}}(v^{\tau_{\boldsymbol{\mathcal{E}}}})}{1-v} + u\mathbf{H}_{\mathbf{x}}(u)\mathbf{K}(u,v).$$

3 Particular cases

In this part, we focus on the particular cases where $\ell = 1$ or $r = 1$. When $\ell = 1$, we are dealing with one ligand which can meet several receptors while in the case $r = 1$, we are concerned by one receptor which can be reached by several ligands.

3.1 Case $\ell = 1$

In this case, our model is a one-dimensional random walk model and we have $\boldsymbol{\mathcal{E}} = \mathcal{R}$. We adapt the general settings by putting

$$T_{n,\mathcal{R}} = \sum_{\iota=1}^{n} \mathbb{1}_{\{S(\iota)\in\mathcal{R}\}},$$

and by writing, for $1 \le i, j \le r$, $\iota \in \mathbb{N}$ and $\jmath \in \mathbb{N}^*$, the following probabilities:

$$\begin{aligned}
p_{i,j}^{(\iota)} &= \mathbb{P}_{a_i}\{S(\iota) = a_j\},\\
q_{i,j}^{(\jmath)} &= \mathbb{P}_{a_i}\{\tau_{\mathcal{R}} = \jmath, S(\tau_{\mathcal{R}}) = a_j\} = \mathbb{P}_{a_i}\{\tau_{\mathcal{R}} = \tau_{a_j} = \jmath\},\\
r_i^{(k,n)} &= \mathbb{P}_{a_i}\{T_{n,\mathcal{R}} = k\}.
\end{aligned}$$

We also introduce the generating functions

$$\begin{aligned}
G_{i,j}(u) &= \sum_{\iota=0}^{\infty} p_{i,j}^{(\iota)} u^\iota,\\
H_{i,j}(u) &= \sum_{\jmath=1}^{\infty} q_{i,j}^{(\jmath)} u^\jmath = \mathbb{E}_{a_i}(u^{\tau_{\mathcal{R}}}, S(\tau_{\mathcal{R}}) = a_j),\\
K_i(u,v) &= \sum_{\substack{k,n\in\mathbb{N}:\\ k\le n}} r_i^{(k,n)} u^k v^n = \sum_{n=0}^{\infty} \mathbb{E}_{a_i}\big(u^{T_{n,\mathcal{R}}}\big) v^n,
\end{aligned}$$

together with the related (finite) matrices

$$\mathbf{G}(u)=\left(G_{i,j}(u)\right)_{1\le i,j\le r},\ \mathbf{H}(u)=\left(H_{i,j}(u)\right)_{1\le i,j\le r},\ \mathbf{K}(u,v)=\left(K_i(u,v)\right)_{1\le i\le r}.$$

Referring to Section 1.4, we explicitly have

$$p_{i,j}^{(\iota)}=\binom{\iota}{\frac{\iota+a_i-a_j}{2}}p^{(\iota+\alpha_j-\alpha_i)/2}q^{(\iota+\alpha_i-\alpha_j)/2}.$$

$$G_{i,j}(u)=\begin{cases}\dfrac{[B^-(u)]^{a_i-a_j}}{A(u)} & \text{if } a_i>a_j,\\ \dfrac{1}{A(u)} & \text{if } a_i=a_j,\\ \dfrac{[B^+(u)]^{a_i-a_j}}{A(u)} & \text{if } a_i<a_j.\end{cases}$$

The matrix $\mathbf{H}(u)$ is three-diagonal. More precisely, we have $H_{i,j}(u)=0$ for $|i-j|\ge 2$ and

$$\begin{aligned}H_{i,i}(u)&=\mathbb{E}_{a_i}\left(u^{\tau_{a_i}},\tau_{a_i}<\tau_{a_{i-1},a_{i+1}}\right),\\ H_{i,i+1}(u)&=\mathbb{E}_{a_i}\left(u^{\tau_{a_{i+1}}},\tau_{a_{i+1}}<\tau_{a_i}\right),\\ H_{i,i-1}(u)&=\mathbb{E}_{a_i}\left(u^{\tau_{a_{i-1}}},\tau_{a_{i-1}}<\tau_{a_i}\right).\end{aligned}$$

These quantities are explicitly given by (1.11), (1.12) and (1.13) in Section 1.4 which explicitly contains the matrix $\mathbf{H}(u)$.

The functions $H_{i,j}(u)$ can be also obtained by equations (2.2) which read here

$$G_{i,j}(u)=\delta_{i,j}+\sum_{k=1}^{r}H_{i,k}(u)G_{k,j}(u).$$

This can be rewritten in terms of matrices as in (2.3), by introducing the unit rxr-matrix $\mathbf{I}_r=\left(\delta_{i,j}\right)_{1\le i,j\le r}$,

$$\mathbf{G}(u)=\mathbf{I}_r+\mathbf{H}(u)\mathbf{G}(u)$$

and then

$$\mathbf{H}(u)=\mathbf{I}_r-\mathbf{G}(u)^{-1}. \tag{3.1}$$

Remark 3.1 An alternative representation of $\mathbf{H}(u)$ can be obtained as follows. Let us introduce the probabilities

$$\rho_{i,j}^{(\jmath)}=\mathbb{P}_{a_i}\{\tau_{a_j}=\jmath\},\quad \tilde{\rho}_{i,j}^{(\jmath)}=\begin{cases}\rho_{i,j}^{(\jmath)} & \text{if } i\ne j\\ \delta_{\jmath,0} & \text{if } i=j\end{cases}$$

and their generating functions

$$L_{i,j}(u)=\sum_{\jmath=1}^{\infty}\rho_{i,j}^{(\jmath)}u^{\jmath}=\mathbb{E}_{a_i}(u^{\tau_{a_j}}),\quad \tilde{L}_{i,j}(u)=\sum_{\jmath=1}^{\infty}\tilde{\rho}_{i,j}^{(\jmath)}u^{\jmath}=\begin{cases}L_{i,j}(u) & \text{if } i\ne j\\ 1 & \text{if } i=j\end{cases}$$

together with their related matrices

$$\mathbf{L}(u) = (L_{i,j}(u))_{1\le i,j\le r}, \quad \tilde{\mathbf{L}}(u) = (\tilde{L}_{i,j}(u))_{1\le i,j\le r}.$$

Actually, the $\tilde{\rho}_{i,j}^{(\jmath)}$'s are associated with time $\tilde{\tau}_{a_i} = \min\{\iota \in \mathbb{N} : S(\iota) = a_i\}$. Notice that τ_{a_i} and $\tilde{\tau}_{a_i}$ coincide when the starting point differs from a_i; in the case where the walk starts at a_i, $\tilde{\tau}_{a_i} = 0$ while τ_{a_i} is the first return time at a_i. We observe that

$$\begin{aligned}\rho_{i,j}^{(\jmath)} &= \mathbb{P}_{a_i}\{\tau_{\mathcal{R}} \le \jmath, \tau_{a_j} = \jmath\}\\ &= \sum_{\iota=1}^{\jmath-1}\sum_{k=1}^{r} \mathbb{P}_{a_i}\{\tau_{\mathcal{R}} = \iota, S(\tau_{\mathcal{R}}) = a_k\}\mathbb{P}_{a_k}\{\tau_{a_j} = \jmath-\iota\}\\ &\quad + \sum_{k=1}^{r}\mathbb{P}_{a_i}\{\tau_{\mathcal{R}} = \jmath, S(\tau_{\mathcal{R}}) = a_j\} = \sum_{\iota=1}^{\jmath-1}\sum_{k=1}^{r} q_{i,k}^{(\iota)}\,\tilde{\rho}_{k,j}^{(\jmath-\iota)}.\end{aligned}$$

This implies the following relationship for the corresponding generating functions:

$$L_{i,j}(u) = \sum_{k=1}^{r} H_{i,k}(u)\tilde{L}_{k,j}(u)$$

or, equivalently,

$$\mathbf{L}(u) = \mathbf{H}(u)\tilde{\mathbf{L}}(u),$$

from which we deduce

$$\mathbf{H}(u) = \mathbf{L}(u)\tilde{\mathbf{L}}(u)^{-1}.$$

As a check, we notice that $\tilde{\mathbf{L}}(u) - \mathbf{L}(u)$ is a diagonal matrix:

$$\tilde{\mathbf{L}}(u) - \mathbf{L}(u) = \operatorname{diag}(1 - L_{i,i}(u))_{1\le i\le r} = \operatorname{diag}(1/G_{i,i}(u))_{1\le i\le r}$$

and then, by (1.4), $\mathbf{G}(u)[\tilde{\mathbf{L}}(u) - \mathbf{L}(u)] = \tilde{\mathbf{L}}(u)$ which entails $\mathbf{L}(u)\tilde{\mathbf{L}}(u)^{-1} = \mathbf{I}_r - \mathbf{G}(u)^{-1}$. ■

Finally, the functions $K_i(v)$, $1 \le i \le r$, are given by equation (2.7) which reads here

$$K_i(u,v) = \frac{1 - \mathbb{E}_{a_i}(v^{\tau_{\mathcal{R}}})}{1-v} + u\sum_{j=1}^{r} H_{i,j}(v)K_j(u,v),$$

and the equivalent matrix equation (2.8) writes

$$[\mathbf{I}_r - u\mathbf{H}(v)]\mathbf{K}(u,v) = \frac{1}{1-v}\,[\mathbb{1}_r - \tilde{\mathbf{H}}(v)].$$

The above column-matrix $\tilde{\mathbf{H}}(v)$ is defined by

$$\tilde{\mathbf{H}}(v) = (\mathbb{E}_{a_i}(v^{\tau_{\mathcal{R}}}))_{1\le i\le r} = \mathbf{H}(v)\mathbb{1}_r.$$

The solution is the finite column-matrix

$$\begin{aligned}\mathbf{K}(u,v) &= \frac{1}{1-v}\,[\mathbf{I}_r - u\mathbf{H}(v)]^{-1}[\mathbb{1}_r - \tilde{\mathbf{H}}(v)]\\ &= \frac{1}{1-v}\,[\mathbf{I}_r - u\mathbf{H}(v)]^{-1}[\mathbf{I}_r - \mathbf{H}(v)]\mathbb{1}_r. \end{aligned} \tag{3.2}$$

Remark 3.2 Referring to Equation (2.9), we see that the modified generating function $\tilde{\mathbf{K}}(u,v)=\mathbf{K}(u,v)-\frac{1}{u(1-v)}\mathbb{1}_r$ is given by

$$\tilde{\mathbf{K}}(u,v)=\frac{u-1}{u(1-v)}\,[\mathbf{I}_r-u\mathbf{H}(v)]^{-1}\mathbb{1}_r. \tag{3.3}$$

■

Remark 3.3 By inserting the expression (3.1) of $\mathbf{H}(v)$ by means of $\mathbf{G}(v)$ into (3.2), we get that

$$\begin{aligned}[\mathbf{I}_r-u\mathbf{H}(v)]^{-1}[\mathbf{I}_r-\mathbf{H}(v)]&=[(1-u)\mathbf{I}_r+u\mathbf{G}(v)^{-1}]^{-1}[\mathbf{G}(v)^{-1}]\\&=[(1-u)\mathbf{G}(v)+u\mathbf{I}_r]^{-1}.\end{aligned}$$

So, $\mathbf{K}(u,v)$ can be expressed in terms of $\mathbf{G}(v)$ as

$$\mathbf{K}(u,v)=\frac{1}{1-v}\,[(1-u)\mathbf{G}(v)+u\mathbf{I}_r]^{-1}\mathbb{1}_r.$$

This representation is simpler than (3.2). Nonetheless, it is not tractable for inverting the generating function $\mathbf{K}(u,v)$ with respect to u. ■

Expanding $[\mathbf{I}_r-u\mathbf{H}(v)]^{-1}$ into $\sum_{k=0}^{\infty}u^k\mathbf{H}(v)^k$, we get by (3.2) (or (3.3))

$$\mathbf{K}(u,v)=\frac{1}{1-v}\sum_{k=0}^{\infty}u^k[\mathbf{H}(v)^k(\mathbf{I}_r-\mathbf{H}(v))\mathbb{1}_r]$$

from which we extract the following proposition.

Proposition 3.4 *The probability distribution of $T_{n,\mathcal{R}}$ satisfies, for any $k\in\mathbb{N}$,*

$$\sum_{n=k}^{\infty}\Big((\mathbb{P}_{a_i}\{T_{n,\mathcal{R}}=k\})_{1\le i\le r}\Big)\,v^n=\frac{1}{1-v}\,\mathbf{H}(v)^k[\mathbf{I}_r-\mathbf{H}(v)]\mathbb{1}_r \tag{3.4}$$

where each sides of the equality are column-matrices.

We could go further in the computations: expanding $1/(1-v)$ into $\sum_{n=0}^{\infty}v^n$, we obtain by (3.4) that

$$\sum_{n=k}^{\infty}\left(\left(r_i^{(k,n)}\right)_{1\le i\le r}\right)v^n=\left(\sum_{n=k}^{\infty}v^n\right)\mathbf{H}(v)^k[\mathbf{I}_r-\mathbf{H}(v)]\mathbb{1}_r.$$

Introducing the matrix $\mathbf{Q}^{(\iota)}=\big(q_{i,j}^{(\iota)}\big)_{1\le i,j\le r}$, we rewrite $\mathbf{H}(v)$ as $\sum_{\iota=1}^{\infty}\mathbf{Q}^{(\iota)}v^{\iota}$. Then,

$$\mathbf{H}(v)^k=\sum_{\iota=k}^{\infty}\left(\sum_{\substack{\iota_1,\dots,\iota_k\in\mathbb{N}^*:\\ \iota_1+\dots+\iota_k=\iota}}\mathbf{Q}^{(\iota_1)}\dots\mathbf{Q}^{(\iota_k)}\right)v^{\iota}$$

and

$$\mathbf{H}(v)^k[\mathbf{I}_r-\mathbf{H}(v)]$$
$$=\sum_{\iota=k}^{\infty}\left(\sum_{\substack{\iota_1,\dots,\iota_k\in\mathbb{N}^*:\\ \iota_1+\dots+\iota_k=\iota}}\mathbf{Q}^{(\iota_1)}\dots\mathbf{Q}^{(\iota_k)}-\sum_{\substack{\iota_1,\dots,\iota_{k+1}\in\mathbb{N}^*:\\ \iota_1+\dots+\iota_{k+1}=\iota}}\mathbf{Q}^{(\iota_1)}\dots\mathbf{Q}^{(\iota_{k+1})}\right)v^{\iota}.$$

Next,

$$\left(\sum_{n=k}^{\infty}v^n\right)\mathbf{H}(v)^k[\mathbf{I}_r-\mathbf{H}(v)]\mathbb{1}_r$$
$$=\sum_{n=k}^{\infty}\left[\sum_{\iota=k}^{n}\left(\sum_{\substack{\iota_1,\dots,\iota_k\in\mathbb{N}^*:\\ \iota_1+\dots+\iota_k=\iota}}\mathbf{Q}^{(\iota_1)}\dots\mathbf{Q}^{(\iota_k)}\mathbb{1}_r-\sum_{\substack{\iota_1,\dots,\iota_{k+1}\in\mathbb{N}^*:\\ \iota_1+\dots+\iota_{k+1}=\iota}}\mathbf{Q}^{(\iota_1)}\dots\mathbf{Q}^{(\iota_{k+1})}\mathbb{1}_r\right)\right]v^{\iota}.$$

We finally deduce the result below.

Theorem 3.5 *The probability $\mathbb{P}_{a_i}\{T_{n,\mathcal{R}}=k\}$ is the i-th term of the column-matrix*

$$\sum_{\substack{\iota_1,\dots,\iota_k\in\mathbb{N}^*:\\ k\le\iota_1+\dots+\iota_k\le n}}\mathbf{Q}^{(\iota_1)}\dots\mathbf{Q}^{(\iota_k)}\mathbb{1}_r-\sum_{\substack{\iota_1,\dots,\iota_{k+1}\in\mathbb{N}^*:\\ k\le\iota_1+\dots+\iota_{k+1}\le n}}\mathbf{Q}^{(\iota_1)}\dots\mathbf{Q}^{(\iota_{k+1})}\mathbb{1}_r.$$

The expectation of $T_{n,\mathcal{R}}$ can be derived by evaluating the derivative of its generating function at $u=1$. Indeed,

$$\begin{aligned}\frac{\partial\mathbf{K}}{\partial u}(1,v)&=\frac{1}{1-v}\,\mathbf{H}(v)[\mathbf{I}_r-u\mathbf{H}(v)]^{-2}[\mathbf{I}_r-\mathbf{H}(v)]\mathbb{1}_r\Big|_{u=1}\\&=\frac{1}{1-v}\,\mathbf{H}(v)[\mathbf{I}_r-\mathbf{H}(v)]^{-1}\mathbb{1}_r.\end{aligned}$$

Since

$$\mathbf{H}(v)[\mathbf{I}_r-\mathbf{H}(v)]^{-1}=[\mathbf{I}_r-(\mathbf{I}_r-\mathbf{H}(v))][\mathbf{I}_r-\mathbf{H}(v)]^{-1}=[\mathbf{I}_r-\mathbf{H}(v)]^{-1}-\mathbf{I}_r,$$

we get the expectation of $T_{n,\mathcal{R}}$ when the random walk starts at a receptor at time 0:

$$(\mathbb{E}_{a_i}(T_{n,\mathcal{R}}))_{1\le i\le r}=\frac{1}{1-v}\left([\mathbf{I}_r-\mathbf{H}(v)]^{-1}-\mathbf{I}_r\right)\mathbb{1}_r.$$

Next, the second moment of $T_{n,\mathcal{R}}$ can be extracted by computing the second derivative of its generating function at $u=1$:

$$\begin{aligned}\frac{\partial^2\mathbf{K}}{\partial u^2}(1,v)&=\frac{2}{1-v}\,\mathbf{H}(v)^2[\mathbf{I}_r-u\mathbf{H}(v)]^{-3}[\mathbf{I}_r-\mathbf{H}(v)]\mathbb{1}_r\Big|_{u=1}\\&=\frac{2}{1-v}\,\mathbf{H}(v)^2[\mathbf{I}_r-\mathbf{H}(v)]^{-2}\mathbb{1}_r.\end{aligned}$$

Since

$$\begin{aligned}\mathbf{H}(v)^2[\mathbf{I}_r-\mathbf{H}(v)]^{-2}+\mathbf{H}(v)[\mathbf{I}_r-\mathbf{H}(v)]^{-1}&\\ &=\mathbf{H}(v)[\mathbf{I}_r-\mathbf{H}(v)]^{-2}[\mathbf{H}(v)+(\mathbf{I}_r-\mathbf{H}(v))]\\ &=[\mathbf{I}_r-(\mathbf{I}_r-\mathbf{H}(v))][\mathbf{I}_r-\mathbf{H}(v)]^{-2}\\ &=[\mathbf{I}_r-\mathbf{H}(v)]^{-2}-[\mathbf{I}_r-\mathbf{H}(v)]^{-1},\end{aligned}$$

we get the variance of $T_{n,\mathcal{R}}$:

$$\left(\mathrm{var}_{a_i}(T_{n,\mathcal{R}})\right)_{1\le i\le r}=\frac{1}{1-v}\left([\mathbf{I}_r-\mathbf{H}(v)]^{-2}-[\mathbf{I}_r-\mathbf{H}(v)]^{-1}\right)\mathbb{1}_r.$$

3.2 Case $r=1$

In this case, we have one receptor: $\mathcal{R}=\{a\}$, and then $\boldsymbol{\mathcal{E}}=\bigcup_{j=1}^{\ell}\boldsymbol{\mathcal{E}}_j$ where $\boldsymbol{\mathcal{E}}_j$ is the hyperplane $\mathbb{Z}^{j-1}\times\{a\}\times\mathbb{Z}^{\ell-j}$. Formulas (2.3) and (2.8) in Theorems 2.3 and 2.4 do not simplify.

3.3 Case $\ell=1$ and $r=2$

We have in this case $\boldsymbol{\mathcal{E}}=\mathcal{R}=\{a_1,a_2\}$. We work with the modified generating function $\tilde{\mathbf{K}}(u,v)$ given by (3.3). We have to invert the 2x2-matrix $\mathbf{I}_2-u\mathbf{H}(v)$. Here, $\mathbf{H}(v)$ reads

$$\mathbf{H}(v)=\begin{pmatrix}H_{1,1}(v) & H_{1,2}(v)\\ H_{2,1}(v) & H_{2,2}(v)\end{pmatrix}=\begin{pmatrix}\mathbb{E}_{a_1}(v^{\tau_{a_1}},\tau_{a_1}<\tau_{a_2}) & \mathbb{E}_{a_1}(v^{\tau_{a_2}},\tau_{a_2}<\tau_{a_1})\\ \mathbb{E}_{a_2}(v^{\tau_{a_1}},\tau_{a_1}<\tau_{a_2}) & \mathbb{E}_{a_2}(v^{\tau_{a_2}},\tau_{a_2}<\tau_{a_1})\end{pmatrix}.$$

The entries of this matrix are given by (1.11) and (1.12). Setting

$$\Delta(u,v)=1-u[H_{1,1}(v)+H_{2,2}(v)]+u^2[H_{1,1}(v)H_{2,2}(v)-H_{1,2}(v)H_{2,1}(v)],$$

we have

$$[\mathbf{I}_2-u\mathbf{H}(v)]^{-1}=\frac{1}{\Delta(u,v)}\begin{pmatrix}1-uH_{2,2}(v) & uH_{2,1}(v)\\ uH_{1,2}(v) & 1-uH_{1,1}(v)\end{pmatrix}$$

and then, by (3.3),

$$\tilde{\mathbf{K}}(u,v)=\frac{u-1}{u(1-v)\Delta(u,v)}\begin{pmatrix}1+u[H_{2,1}(v)-H_{2,2}(v)]\\ 1-u[H_{1,1}(v)-H_{1,2}(v)]\end{pmatrix}.$$

3.4 Case $\ell=r=1$

We have in this case $\boldsymbol{\mathcal{E}}=\mathcal{R}=\{a\}$. The sojourn time of interest reads now

$$T_{n,a}=\sum_{\iota=1}^{n}\mathbb{1}_{\{S(\iota)=a\}}.$$

We are dealing with the local time of the one-dimensional random walk at a. We adapt the settings by putting

$$\begin{aligned} p^{(\iota)} &= \mathbb{P}_a\{S(\iota)=a\} = \mathbb{P}_0\{S(\iota)=0\},\\ q^{(\jmath)} &= \mathbb{P}_a\{\tau_a=\jmath\} = \mathbb{P}_0\{\tau_0=\jmath\},\\ r^{(k,n)} &= \mathbb{P}_a\{T_{n,a}=k\} = \mathbb{P}_0\{T_{n,0}=k\}, \end{aligned}$$

and

$$G(u)=\sum_{\iota=0}^{\infty} p^{(\iota)}u^{\iota},\quad H(u)=\sum_{\jmath=1}^{\infty} q^{(\jmath)}u^{\jmath}=\mathbb{E}_0(u^{\tau_0}).$$

We explicitly have

$$p^{(\iota)}=\binom{\iota}{\iota/2}(pq)^{\iota/2},\quad q^{(\jmath)}=\frac{1}{\jmath-1}\binom{\jmath}{\jmath/2}(pq)^{\jmath/2},$$

$$G(u)=\frac{1}{A(u)},\quad H(u)=1-\frac{1}{G(u)}=1-A(u).$$

By (3.4), we have

$$\sum_{n=k}^{\infty} r^{(k,n)}v^n=\frac{A(v)}{1-v}\,[1-A(v)]^k.$$

In this case, we can explicitly invert the previous generating function. Indeed, by invoking (1.2), we get

$$\begin{aligned} \sum_{n=k}^{\infty} r^{(k,n)}v^n &= (1-4pqv^2)\left(\sum_{l=0}^{\infty} v^l\right)\sum_{m=0}^{\infty} 2^k\binom{2m+k}{m}(pqv^2)^{m+k}\\ &= 2^k\left(1+v+(1-4pq)\sum_{l=2}^{\infty} v^l\right)\sum_{m=k}^{\infty}\binom{2m-k}{m}(pq)^m v^{2m}. \end{aligned}$$

The foregoing double sum can be easily transformed as follows:

$$\sum_{l=2}^{\infty} v^l\sum_{m=k}^{\infty}\binom{2m-k}{m}(pq)^m v^{2m}=\sum_{n=2k+2}^{\infty}\left(\sum_{m=k}^{[n/2]-1}\binom{2m-k}{m}(pq)^m\right)v^n$$

and then

$$\begin{aligned} \sum_{n=k}^{\infty} r^{(k,n)}v^n = 2^k\Bigg[&\sum_{m=k}^{\infty}\binom{2m-k}{m}(pq)^m v^{2m}+\sum_{m=k}^{\infty}\binom{2m-k}{m}(pq)^m v^{2m+1}\\ &+(1-4pq)\sum_{n=2k+2}^{\infty}\left(\sum_{m=k}^{[n/2]-1}\binom{2m-k}{m}(pq)^m\right)v^n\Bigg]. \end{aligned}$$

Hence, by identifying the terms in v^n, we find that for even n

$$r^{(k,n)}=2^k\left[\binom{n-k}{n/2}(pq)^{n/2}+(1-4pq)\sum_{m=k}^{n/2-1}\binom{2m-k}{m}(pq)^m\right] \tag{3.5}$$

and for odd n, observing that $r^{(k,n)} = r^{(k,n-1)}$ since the last step of the random walk cannot vanish in this case,

$$r^{(k,n)} = 2^k \left[\binom{n-k-1}{(n-1)/2} (pq)^{(n-1)/2} + (1-4pq) \sum_{m=k}^{(n-3)/2} \binom{2m-k}{m} (pq)^m \right].$$

In the two above formulas we adopted the convention that $\sum_{m=k}^{l} = 0$ when $k > l$. At this stage, we see that in the particular case of the symmetric random walk (corresponding to $p = q = \frac{1}{2}$), the foregoing formulas simply reduce to

$$r^{(k,n)} = \begin{cases} \dfrac{\binom{n-k}{n/2}}{2^{n-k}} & \text{if } n \text{ is even,} \\[2ex] \dfrac{\binom{n-k-1}{(n-1)/2}}{2^{n-k-1}} & \text{if } n \text{ is odd.} \end{cases}$$

Going back to (3.5), we get for even n:

$$\begin{aligned} r^{(k,n)} &= 2^k \left[\binom{n-k}{n/2} (pq)^{n/2} + \sum_{m=k}^{n/2-1} \binom{2m-k}{m} (pq)^m - 4 \sum_{m=k}^{n/2-1} \binom{2m-k}{m} (pq)^{m+1} \right] \\ &= 2^k \left[\binom{n-k}{n/2} (pq)^{n/2} + \sum_{m=k}^{n/2-1} \binom{2m-k}{m} (pq)^m - 4 \sum_{m=k+1}^{n/2} \binom{2m-k-2}{m-1} (pq)^m \right] \\ &= 2^k \left[(pq)^k + \sum_{m=k+1}^{n/2} \left[\binom{2m-k}{m} - 4 \binom{2m-k-2}{m-1} \right] (pq)^m \right] \\ &= 2^k \left[(pq)^k + \sum_{m=k+1}^{n/2} \frac{(2m-k-2)!}{m!(m-k)!} (k^2+k-2m)(pq)^m \right] \\ &= (2pq)^k \left[1 + \sum_{m=1}^{n/2-k} \frac{(2m+k-2)!}{m!(m+k)!} (k^2-k-2m)(pq)^m \right]. \end{aligned}$$

Consequently, we have obtained the following result.

Theorem 3.6 *The probability distribution of $T_{n,0}$ is given, for even n, by*

$$\mathbb{P}_0\{T_{n,0} = k\} = (2pq)^k \left[1 + \sum_{m=1}^{n/2-k} \frac{(2m+k-2)!}{m!(m+k)!} (k^2-k-2m)(pq)^m \right]$$

and for odd n by $\mathbb{P}_0\{T_{n,0} = k\} = \mathbb{P}_0\{T_{n-1,0} = k\}$. When $p = q = \frac{1}{2}$ (case of the symmetric random walk), this distribution reduces, for even n, to

$$\mathbb{P}_0\{T_{n,0} = k\} = \frac{\binom{n-k}{n/2}}{2^{n-k}}.$$

In particular, we have

$$r^{0,n} = 1 - \sum_{m=1}^{n/2} \frac{\binom{2m}{m}}{2m-1} (pq)^m.$$

This quantity, which represents the probability $\mathbb{P}_0\{T_{n,0} = 0\}$ is nothing but the probability $\mathbb{P}_0\{\tau_0 > n\}$.

4 Further investigations

In a more realistic model, the plasmic membrane should be viewed as a closed curve and its roaming constituents should be modeled for instance by random walks on the finite torus $\mathbb{Z}/N\mathbb{Z} = \{0, 1, \dots, N-1\}$ for a fixed possibly large integer N (with the usual rule $N \equiv 0$ $[\mathrm{mod}\, N]$) or by Brownian motions on the continuous torus $\mathbb{R}/\mathbb{Z}$.

Moreover, the rafts could be built by aggregating several receptors, that is by choosing several sequences of successive receptors: $\{a_1, a_1+1, \dots, a_1+l_1-1\}$, $\{a_2, a_2+1, \dots, a_2+l_2-1\}, \dots, \{a_r, a_r+1, \dots, a_r+l_r-1\}$ with $a_1 + l_1 \leq a_2$, $a_2 + l_2 \leq a_3, \dots, a_{r-1} + l_{r-1} \leq a_r$, where the $l_1, \dots, l_r$ are the length of the rafts. The effect of this kind of non-homogeneous repartition of receptors on the binding distribution could provide a possible functional property of rafts. In addition, the localizations and the lengths of the rafts could be random and then we should also consider that the numbers $a_1, \dots, a_r$ and $l_1, \dots, l_r$ are random variables.

So far we concentrate on diffusion-limited reaction neglecting the binding duration. But in the reality, ligand-receptor binding induces some delay in the mechanism, that is when a ligand meets a receptor, they bind during a certain (possibly random) amount of time before unbinding. So, we should also introduce convenient delayed random walk or Brownian motion for recreating the real biological process.

In Subsection 4.1, we give some information about a possible random walk model on the torus $\mathbb{Z}/N\mathbb{Z}$ and in Subsection 4.2, we address possible continuous models involving Brownian motion on the line $\mathbb{R}$ or on the torus $\mathbb{R}/\mathbb{Z}$.

4.1 A random walk model on $\mathbb{Z}/N\mathbb{Z}$

For building a model on $\mathbb{Z}/N\mathbb{Z}$ with a deterministic set of receptors $\mathcal{R}' = \{a_1, \dots, a_r\}$, we introduce an ℓ–dimensional random walk $(\mathbf{S}'(\iota))_{\iota\in\mathbb{N}}$ on the finite set $(\mathbb{Z}/N\mathbb{Z})^\ell$. Set

$$\boldsymbol{\mathcal{E}}' = \bigcup_{j=1}^{\ell} \left[(\mathbb{Z}/N\mathbb{Z})^{j-1} \times \mathcal{R}' \times (\mathbb{Z}/N\mathbb{Z})^{\ell-j} \right].$$

This set is finite: $\#\boldsymbol{\mathcal{E}}' = \ell r N^{\ell-1}$, and the sojourn time of interest writes here

$$\mathbf{T}'_{n,\boldsymbol{\mathcal{E}}'} = \sum_{\iota=1}^{n} \mathbb{1}_{\{\mathbf{S}'(\iota)\in\boldsymbol{\mathcal{E}}'\}}.$$

For tackling the computation of the probability distribution of $\mathbf{T}'_{n,\boldsymbol{\mathcal{E}}'}$, we observe that we can pass from the one–dimensional random walk $(S'(\iota))_{\iota\in\mathbb{N}}$ on $\mathbb{Z}/N\mathbb{Z}$ to a walk on $\mathbb{Z}$ in the following manner: set, for $\iota \in \mathbb{N}$,

$$S(\iota) = S'(\iota) + \alpha_\iota N$$

where α_ι is the number of upcrossings of level 0 (i.e. times σ such that $S'(\sigma-1) = N-1, S'(\sigma) = 0, S'(\sigma+1) = 1$) minus that of downcrossing of 0 (i.e. times σ such that $S'(\sigma-1) = 1, S'(\sigma) = 0, S'(\sigma+1) = N-1$) up to time ι. Then, $(S(\iota))_{\iota\in\mathbb{N}}$ is a random walk on $\mathbb{Z}$. Conversely, $S'(\iota) = S(\iota) \mod N = S(\iota) - N[S(\iota)/N]$. Notice that in our new setting, we mark with primes all quantities related to the walks on $\mathbb{Z}/N\mathbb{Z}$ and $(\mathbb{Z}/N\mathbb{Z})^\ell$, while those related to the walks on $\mathbb{Z}$ and $\mathbb{Z}^\ell$ are not marked with any prime.

In this correspondence, the set of receptors $\mathcal{R}'$ in $\mathbb{Z}/N\mathbb{Z}$ becomes an infinite set $\mathcal{R} = \mathcal{R}' + N\mathbb{Z} = \{a_1 + mN, \dots, a_r + mN; m \in \mathbb{Z}\}$ of receptors in $\mathbb{Z}$ and the subset $\boldsymbol{\mathcal{E}}'$ of $(\mathbb{Z}/N\mathbb{Z})^\ell$ becomes the subset of $\mathbb{Z}^\ell$

$$\boldsymbol{\mathcal{E}} = \bigcup_{j=1}^{\ell} \left(\mathbb{Z}^{j-1} \times \mathcal{R} \times \mathbb{Z}^{\ell-j}\right). \tag{4.1}$$

Of course,

$$\mathbf{T}'_{n,\boldsymbol{\mathcal{E}}'} = \sum_{\iota=1}^{n} 1\!\!1_{\{\mathbf{S}(\iota)\in\boldsymbol{\mathcal{E}}\}}.$$

The analysis done in the case of the walk on $\mathbb{Z}^\ell$ (associated with the set (4.1) now) can be carried out for the walk on $(\mathbb{Z}/N\mathbb{Z})^\ell$ exactly in the same way *mutatis mutandis*: the generating matrix $\mathbf{K}'(u,v)$ of the family of numbers $\mathbf{P}_\mathbf{x}\{\mathbf{T}'_{n,\boldsymbol{\mathcal{E}}'} = k\}$, $\mathbf{x} \in \boldsymbol{\mathcal{E}}'$, $k, n \in \mathbb{N}$, is given by

$$\mathbf{K}'(u,v) = \frac{1}{1-v}\,[\mathbf{I}_{\boldsymbol{\mathcal{E}}'} - u\mathbf{H}'(v)]^{-1}[\mathbf{I}_{\boldsymbol{\mathcal{E}}'} - \mathbf{H}'(v)]1\!\!1_{\boldsymbol{\mathcal{E}}'}$$

where $\mathbf{I}_{\boldsymbol{\mathcal{E}}'}$ is the identity matrix on $\boldsymbol{\mathcal{E}}'$: $\mathbf{I}_{\boldsymbol{\mathcal{E}}'} = (\delta_{\mathbf{x},\mathbf{y}})_{\mathbf{x},\mathbf{y}\in\boldsymbol{\mathcal{E}}'}$, $1\!\!1_{\boldsymbol{\mathcal{E}}'}$ is the column-matrix consisting of 1, that is $(1)_{\mathbf{x}\in\boldsymbol{\mathcal{E}}'}$, and

$$\mathbf{H}'(v) = \mathbf{I}_{\boldsymbol{\mathcal{E}}'} - \mathbf{G}'(v)^{-1}.$$

In the foregoing definition of $\mathbf{H}'(v)$, $\mathbf{G}'(v)$ is the matrix generating function of the probabilities $\mathbf{P}'_\mathbf{x}\{\mathbf{S}'(\iota) = \mathbf{y}\}$, $\mathbf{x}, \mathbf{y} \in \boldsymbol{\mathcal{E}}'$ which are given by

$$\mathbf{P}'_\mathbf{x}\{\mathbf{S}'(\iota) = \mathbf{y}\} = \prod_{j=1}^{\ell} \mathbb{P}'_{x_j}\{S'(\iota) = y_j\}.$$

The probabilities related to the case $\ell = 1$ can be evaluated as follows. For $x, y \in \mathbb{Z}/N\mathbb{Z}$,

$$\begin{aligned}
\mathbb{P}'_x\{S'(\iota) = y\} &= \sum_{k\in\mathbb{Z}} \mathbb{P}_x\{S(\iota) = y + kN\} \\
&= \sum_{\substack{k\in\mathbb{Z}:\\ (x-y-\iota)/N \le k \le (x-y+\iota)/N}} \binom{\iota}{\frac{\iota+y-x+kN}{2}} p^{(\iota+y-x+kN)/2} q^{(\iota+x-y-kN)/2}.
\end{aligned}$$

Let us have a look to the case $\ell = r = 1$. For this we introduce the following families of hitting times related to $(S(\iota))_{\iota\in\mathbb{N}}$: for $a, b, c \in \mathbb{Z}$ such that $a < b < c$,

$$\tau_{a,b} = \min\{\iota \geq 1 : S(\iota) \in \{a, b\}\},$$
$$\tau_{a,b,c} = \min\{\iota \geq 1 : S(\iota) \in \{a, b, c\}\},$$

and that related to $(S'(\iota))_{\iota\in\mathbb{N}}$: for $a \in \mathbb{Z}/N\mathbb{Z}$,

$$\tau'_a = \min\{\iota \geq 1 : S'(\iota) = a\}.$$

It is clear that, if the starting point of $(S'(\iota))_{\iota\in\mathbb{N}}$ is a,

$$\tau'_a = \tau_{a-N,a,a+N}.$$

Because of this relationship, the probability distribution of time τ'_a can be explicitly written out. In the same way as (1.13), we have

$$H'(v) = \mathbb{E}'_a\big(v^{\tau'_a}\big) = \mathbb{E}'_0\big(v^{\tau'_0}\big) = \mathbb{E}_0(v^{\tau_{-N,0,N}}) = pv\,\mathbb{E}_1(v^{\tau_{0,N}}) + qv\,\mathbb{E}_{-1}(v^{\tau_{-N,0}}).$$

Invoking (1.9) and the fact that $B^+(v)B^-(v) = q/p$, we obtain

$$\begin{aligned}\mathbb{E}_1(v^{\tau_{0,N}}) &= \frac{(1-B^-(v)^N)B^+(v) - (1-B^+(v)^N)B^-(v)}{B^+(v)^N - B^-(v)^N} \\ &= \frac{B^+(v) - B^-(v)}{B^+(v)^N - B^-(v)^N} + \frac{q}{p}\,\frac{B^+(v)^{N-1} - B^-(v)^{N-1}}{B^+(v)^N - B^-(v)^N},\end{aligned}$$

$$\begin{aligned}\mathbb{E}_{-1}(v^{\tau_{0,N}}) &= \frac{(1-B^-(v)^N)B^+(v)^{N-1} - (1-B^+(v)^N)B^-(v)^{N-1}}{B^+(v)^N - B^-(v)^N} \\ &= \frac{B^+(v)^{N-1} - B^-(v)^{N-1}}{B^+(v)^N - B^-(v)^N} + \left(\frac{q}{p}\right)^{N-1}\frac{B^+(v) - B^-(v)}{B^+(v)^N - B^-(v)^N},\end{aligned}$$

which entails

$$H'(v) = \left[\left(\frac{q}{p}\right)^N + 1\right] pv\,\frac{B^+(v) - B^-(v)}{B^+(v)^N - B^-(v)^N} + 2qv\,\frac{B^+(v)^{N-1} - B^-(v)^{N-1}}{B^+(v)^N - B^-(v)^N}.$$

Finally, the probability distribution of the sojourn time $\mathbf{T}'_{n,0}$ is characterized by the generating function, as in (3.4),

$$\sum_{n=k}^{\infty} \mathbb{P}'_0\{\mathbf{T}'_{n,0} = k\}v^n = \frac{1}{1-v}\,[H'(v)]^k[1 - H'(v)].$$

It seems difficult to extract the probabilities $\mathbb{P}'_0\{\mathbf{T}'_{n,0} = k\}$, $1 \leq k \leq n$, from this last identity.

4.2 Brownian models on $\mathbb{R}$ and $\mathbb{R}/\mathbb{Z}$

At last, we evoke the continuous counterpart to our model in the one-dimensional case. Analogous continuous models hinge on linear Brownian motion (that is the membrane is viewed as the real line $\mathbb{R}$) and circular Brownian motion (in this case, the membrane is modeled as the torus $\mathbb{R}/\mathbb{Z}$). In both cases, the rafts (clustered receptors) are viewed as intervals $[a_1,b_1],\dots,[a_r,b_r]$ of $\mathbb{R}$ or $\mathbb{R}/\mathbb{Z}$, and ligands move like independent Brownian motions $(B_1(s))_{s\geq 0},\dots,(B_\ell(s))_{s\geq 0}$ on $\mathbb{R}$ or $\mathbb{R}/\mathbb{Z}$. Set

$$\mathbf{B}(s)=(B_1(s),\dots,B_\ell(s)) \text{ for any } s\geq 0,$$

and

$$\mathcal{R}=\bigcup_{i=1}^{r}[a_i,b_i],\ \boldsymbol{\mathcal{E}}=\bigcup_{j=1}^{\ell}\left[\mathbb{R}^{j-1}\times\mathcal{R}\times\mathbb{R}^{\ell-j}\right] \text{ or } \bigcup_{j=1}^{\ell}\left[(\mathbb{R}/\mathbb{Z})^{j-1}\times\mathcal{R}\times(\mathbb{R}/\mathbb{Z})^{\ell-j}\right].$$

The process $(\mathbf{B}(s))_{s\geq 0}$ is a Brownian motion on $\mathbb{R}^\ell$ or $(\mathbb{R}/\mathbb{Z})^\ell$. The time that ligands bind with rafts up to a fixed time t is given by the sojourn time of $(\mathbf{B}(s))_{s\geq 0}$ in $\boldsymbol{\mathcal{E}}$:

$$\mathbf{T}_{t,\boldsymbol{\mathcal{E}}}=\int_0^t \mathbb{1}_{\{\mathbf{B}(s)\in\boldsymbol{\mathcal{E}}\}}\,\mathrm{d}s.$$

In the case of linear Brownian motion ($\ell=1$), the computation of the probability distribution of $\mathbf{T}_{t,\boldsymbol{\mathcal{E}}}$ will be the object of a forthcoming work ([4]).

Appendix

A Distribution of the first hitting time for the random walk

The generating function (1.7) writes

$$H^+_{x,a,b}(u)=(2pu)^{b-x}\frac{[1+A(u)]^{x-a}-[1-A(u)]^{x-a}}{[1+A(u)]^{b-a}-[1-A(u)]^{b-a}}$$

Let us introduce the rational fraction defined for $x,y\in\mathbb{N}$ such that $x<y$, by

$$f_{x,y}(\zeta)=\frac{(1+\zeta)^x-(1-\zeta)^x}{(1+\zeta)^y-(1-\zeta)^y}=\frac{\sum_{m=0}^{[(x-1)/2]}\binom{x}{2m+1}\zeta^{2m}}{\sum_{m=0}^{[(y-1)/2]}\binom{y}{2m+1}\zeta^{2m}}.$$

We want to decompose this fraction into partial fractions. The poles of $f_{x,y}$ are the roots of the polynomial $\sum_{m=0}^{[(y-1)/2]}\binom{y}{2m+1}\zeta^{2m}$ which are those of $(1+\zeta)^y-(1-\zeta)^y$ except for 0. It is easily seen that they consist of the family of numbers $\zeta_l=\frac{e^{2il\pi/y}-1}{e^{2il\pi/y}+1}=i\tan(l\pi/y)$ for $1\leq l\leq y-1$ such that $l\neq y/2$. Moreover, the function $\zeta\mapsto f_{x,y}(\zeta)$ is even, so we decompose it as

$$f_{x,y}(\zeta)=\sum_{\substack{l\in\mathbb{N}:\\ 1\leq l<y/2}}\frac{\alpha_l}{\zeta^2-\zeta_l^2}$$

with $\alpha_l = \lim_{\zeta\to\zeta_l}(\zeta^2-\zeta_l^2)f(\zeta)$. We have

$$\alpha_l = \frac{(1+\zeta_l)^x-(1-\zeta_l)^x}{\lim_{\zeta\to\zeta_l}\frac{(1+\zeta)^y-(1-\zeta)^y}{\zeta^2-\zeta_l^2}} = \frac{2\zeta_l}{y}\,\frac{(1+\zeta_l)^x-(1-\zeta_l)^x}{(1+\zeta_l)^{y-1}+(1-\zeta_l)^{y-1}}$$
$$= 2(-1)^{l-1}\cos^{y-x-3}\!\left(\frac{l\pi}{y}\right)\sin\!\left(\frac{l\pi}{y}\right)\sin\!\left(\frac{lx\pi}{y}\right).$$

Therefore,

$$f_{x,y}(\zeta) = 2\sum_{\substack{l\in\mathbb{N}:\\ 1\le l<y/2}}(-1)^{l-1}\frac{\cos^{y-x-3}\big(\frac{l\pi}{y}\big)\sin\big(\frac{l\pi}{y}\big)\sin\big(\frac{lx\pi}{y}\big)}{\zeta^2+\tan^2\big(\frac{l\pi}{y}\big)}.$$

Let us apply this formula to $H^+_{x,a,b}(u) = (2pu)^{b-x}f_{x-a,b-a}(A(u))$. Since

$$A(u)^2+\tan^2\!\left(\frac{l\pi}{b-a}\right) = \frac{1}{\cos^2\big(\frac{l\pi}{b-a}\big)}\left[1-4pq\left(\cos^2\!\left(\frac{l\pi}{b-a}\right)\right)u^2\right],$$

we get that

$$H^+_{x,a,b}(u) = 2(2pu)^{b-x}\sum_{\substack{l\in\mathbb{N}:\\ 1\le l<(b-a)/2}}\frac{\cos^{b-x-1}\big(\frac{l\pi}{b-a}\big)\sin\big(\frac{l\pi}{b-a}\big)\sin\big(\frac{l(b-x)\pi}{b-a}\big)}{1-4pq\big[\cos^2\big(\frac{l\pi}{b-a}\big)\big]u^2}.$$

Using the expansion

$$\frac{1}{1-4pq\left[\cos^2\big(\frac{l\pi}{b-a}\big)\right]u^2} = \sum_{\jmath=0}^{\infty}\left[4pq\cos^2\!\left(\frac{l\pi}{b-a}\right)\right]^{\jmath}u^{2\jmath},$$

we find

$$H^+_{x,a,b}(u) = \sum_{\jmath=0}^{\infty}2(2p)^{b-x}\Bigg[\sum_{\substack{l\in\mathbb{N}:\\ 1\le l<(b-a)/2}}\cos^{b-x-1}\!\left(\frac{l\pi}{b-a}\right)\sin\!\left(\frac{l\pi}{b-a}\right)$$
$$\sin\!\left(\frac{l(b-x)\pi}{b-a}\right)\left(4pq\cos^2\!\left(\frac{l\pi}{b-a}\right)\right)^{\jmath}\Bigg]u^{2\jmath+b-x}.$$

Performing the change of index $\jmath\mapsto(\jmath+x-b)/2$ into the above sum, we obtain

$$H^+_{x,a,b}(u) = \sum_{\jmath=b-x}^{\infty}2\left(\frac{p}{q}\right)^{(x-b)/2}\Bigg[\sum_{\substack{l\in\mathbb{N}:\\ 1\le l<(b-a)/2}}\cos^{b-x-1}\!\left(\frac{l\pi}{b-a}\right)\sin\!\left(\frac{l\pi}{b-a}\right)$$
$$\sin\!\left(\frac{l(b-x)\pi}{b-a}\right)\left(2\sqrt{pq}\cos\!\left(\frac{l\pi}{b-a}\right)\right)^{\jmath}\Bigg]u^{\jmath}$$

from which we finally derive the expression (1.10) of the probability $q^{(\jmath)+}_{x,a,b}$. In a quite analogous way, we can deduce the probability $q^{(\jmath)-}_{x,a,b}$ from $H^-_{x,a,b}(u) = (2qu)^{x-a}f_{b-x,b-a}(A(u))$.

B Distribution of the stopped random walk

As mentioned in Section 1.4.2, the stopping-probabilities can be evaluated by invoking the famous reflection principle. We provide the details in this Appendix.

One-sided threshold

Our aim is to compute the probability of the random walk stopped when hitting the threshold a: $\mathbb{P}_x\{S(\jmath)=y,\jmath\le\tau_a\}$. We begin by first evaluating $\mathbb{P}_x\{S(\jmath)=y,\tau_a<\jmath\}$. We have, for $x,y\in\mathbb{Z}$ such that $x,y<a$ or $x,y>a$,

$$\mathbb{P}_x\{S(\jmath)=y,\tau_a<\jmath\}=\sum_{\iota=1}^{\jmath-1}\mathbb{P}_x\{\tau_a=\iota\}\mathbb{P}_a\{S(\jmath-\iota)=y\}.$$

The reflection principle stipulates that

$$\mathbb{P}_a\{S(\jmath-\iota)=y\}=\left(\frac{p}{q}\right)^{y-a}\mathbb{P}_a\{S(\jmath-\iota)=2a-y\}.$$

Hence,

$$\begin{aligned}\mathbb{P}_x\{S(\jmath)=y,\tau_a<\jmath\}&=\sum_{\iota=1}^{\jmath-1}\mathbb{P}_x\{\tau_a=\iota\}\left(\frac{p}{q}\right)^{y-a}\mathbb{P}_a\{S(\jmath-\iota)=2a-y\}\\&=\left(\frac{p}{q}\right)^{y-a}\mathbb{P}_x\{S(\jmath)=2a-y,\tau_a<\jmath\}\\&=\left(\frac{p}{q}\right)^{y-a}\mathbb{P}_x\{S(\jmath)=2a-y\}.\end{aligned}$$

In the last equality, we used the fact that x and $2a-y$ are around a and then the condition $\tau_a<\jmath$ is redundant. Therefore,

$$\mathbb{P}_x\{S(\jmath)=y,\jmath\le\tau_a\}=\mathbb{P}_x\{S(\jmath)=y\}-\left(\frac{p}{q}\right)^{y-a}\mathbb{P}_x\{S(\jmath)=2a-y\}$$

which is nothing but (1.14).

Two-sided threshold

Here, we aim to compute the probability of the random walk stopped when hitting one of the two thresholds a and b: $\mathbb{P}_x\{S(\jmath)=y,\jmath\le\tau_{a,b}\}$ under the assumption $x\in(a,b)$. For this, we introduce the successive hitting times of levels a and b (with the usual convention $\min\emptyset=+\infty$):

$$\begin{aligned}&\sigma_0=\tau_a,\quad \sigma_{2l+1}=\min\{\jmath>\sigma_{2l}:S(\jmath)=b\},\quad \sigma_{2l+2}=\min\{\jmath>\sigma_{2l+1}:S(\jmath)=a\},\\&\varsigma_0=\tau_b,\quad \varsigma_{2l+1}=\min\{\jmath>\varsigma_{2l}:S(\jmath)=a\},\quad \varsigma_{2l+2}=\min\{\jmath>\varsigma_{2l+1}:S(\jmath)=b\}.\end{aligned}$$

Notice that if $\tau_a<\tau_b$, then $\sigma_{l+1}=\varsigma_l$ and if $\tau_a>\tau_b$, then $\varsigma_{l+1}=\sigma_l$. In both cases, we see that

$$\min(\sigma_l,\varsigma_l)=\begin{cases}\sigma_l & \text{if } \tau_a<\tau_b\\ \varsigma_l & \text{if } \tau_a>\tau_b\end{cases},\quad \max(\sigma_l,\varsigma_l)=\begin{cases}\sigma_{l+1} & \text{if } \tau_a<\tau_b\\ \varsigma_{l+1} & \text{if } \tau_a>\tau_b\end{cases}$$

and that

$$\max(\sigma_l, \varsigma_l) = \min(\sigma_{l+1}, \varsigma_{l+1}).$$

With these properties at hands, we write the event $\{\tau_{a,b} < \jmath\}$ as follows:

$$\begin{aligned}
\{\tau_{a,b} < \jmath\} &= \{\tau_a < \tau_b, \tau_a < \jmath\} \cup \{\tau_a > \tau_b, \tau_b < \jmath\} \\
&= \left(\bigcup_{l\in\mathbb{N}} \{\tau_a < \tau_b, \sigma_{2l} < \jmath \le \sigma_{2l+2}\}\right) \cup \left(\bigcup_{l\in\mathbb{N}} \{\tau_a > \tau_b, \varsigma_{2l} < \jmath \le \varsigma_{2l+2}\}\right) \\
&= \left(\bigcup_{l\in\mathbb{N}} \{\tau_a < \tau_b, \sigma_{2l} < \jmath \le \varsigma_{2l+1}\}\right) \cup \left(\bigcup_{l\in\mathbb{N}} \{\tau_a > \tau_b, \varsigma_{2l} < \jmath \le \sigma_{2l+1}\}\right) \\
&= \bigcup_{l\in\mathbb{N}} \{\min(\sigma_{2l}, \varsigma_{2l}) < \jmath \le \max(\sigma_{2l+1}, \varsigma_{2l+1})\}.
\end{aligned}$$

Now, the probability of the stopped random walk can be written as

$$\begin{aligned}
&\mathbb{P}_x\{S(\jmath) = y, \jmath \le \tau_{a,b}\} \\
&\quad = \mathbb{P}_x\{S(\jmath) = y\} - \sum_{l=0}^{\infty} \mathbb{P}_x\{S(\jmath) = y, \min(\sigma_{2l}, \varsigma_{2l}) < \jmath \le \max(\sigma_{2l+1}, \varsigma_{2l+1})\}.
\end{aligned} \tag{B.1}$$

Let us evaluate the term lying within the sum (B.1):

$$\begin{aligned}
&\mathbb{P}_x\{S(\jmath) = y, \min(\sigma_{2l}, \varsigma_{2l}) < \jmath \le \max(\sigma_{2l+1}, \varsigma_{2l+1})\} \\
&\qquad = \mathbb{P}_x\{S(\jmath) = y, \min(\sigma_{2l}, \varsigma_{2l}) < \jmath\} - \mathbb{P}_x\{S(\jmath) = y, \max(\sigma_{2l+1}, \varsigma_{2l+1}) < \jmath\} \\
&\qquad = \mathbb{P}_x\{(S(\jmath) = y, \sigma_{2l} < \jmath) \cup (S(\jmath) = y, \varsigma_{2l} < \jmath)\} \\
&\qquad\quad - \mathbb{P}_x\{(S(\jmath) = y, \sigma_{2l+1} < \jmath) \cap (S(\jmath) = y, \varsigma_{2l+1} < \jmath)\} \\
&\qquad = \mathbb{P}_x\{S(\jmath) = y, \sigma_{2l} < \jmath\} + \mathbb{P}_x\{S(\jmath) = y, \varsigma_{2l} < \jmath\} \\
&\qquad\quad - \mathbb{P}_x\{(S(\jmath) = y, \sigma_{2l} < \jmath) \cap (S(\jmath) = y, \varsigma_{2l} < \jmath)\} \\
&\qquad\quad - \mathbb{P}_x\{S(\jmath) = y, \sigma_{2l+1} < \jmath\} - \mathbb{P}_x\{S(\jmath) = y, \varsigma_{2l+1} < \jmath\} \\
&\qquad\quad + \mathbb{P}_x\{(S(\jmath) = y, \sigma_{2l+1} < \jmath) \cup (S(\jmath) = y, \varsigma_{2l+1} < \jmath)\}.
\end{aligned} \tag{B.2}$$

We observe that

$$\begin{aligned}
&\mathbb{P}_x\{(S(\jmath) = y, \sigma_{2l+1} < \jmath) \cup (S(\jmath) = y, \varsigma_{2l+1} < \jmath)\} \\
&\qquad\qquad = \mathbb{P}_x\{S(\jmath) = y, \min(\sigma_{2l+1}, \varsigma_{2l+1}) < \jmath\} \\
&\qquad\qquad = \mathbb{P}_x\{S(\jmath) = y, \max(\sigma_{2l}, \varsigma_{2l}) < \jmath\} \\
&\qquad\qquad = \mathbb{P}_x\{(S(\jmath) = y, \sigma_{2l} < \jmath) \cap (S(\jmath) = y, \varsigma_{2l} < \jmath)\}.
\end{aligned}$$

Thus, two terms of the sum (B.2) cancel and it remains

$$\begin{aligned}
&\mathbb{P}_x\{S(\jmath) = y, \min(\sigma_{2l}, \varsigma_{2l}) < \jmath \le \max(\sigma_{2l+1}, \varsigma_{2l+1})\} \\
&\qquad\qquad = \mathbb{P}_x\{S(\jmath) = y, \sigma_{2l} < \jmath\} + \mathbb{P}_x\{S(\jmath) = y, \varsigma_{2l} < \jmath\} \\
&\qquad\qquad\quad - \mathbb{P}_x\{S(\jmath) = y, \sigma_{2l+1} < \jmath\} - \mathbb{P}_x\{S(\jmath) = y, \varsigma_{2l+1} < \jmath\}.
\end{aligned} \tag{B.3}$$

By using the principle of reflection with respect to level a, we get for the first term of the sum (B.3):

$$
\begin{aligned}
\mathbb{P}_x\{S(\jmath)=y,\sigma_{2l}<\jmath\} &= \sum_{\iota=1}^{\jmath-1}\mathbb{P}_x\{\sigma_{2l}=\iota\}\mathbb{P}_a\{S(\jmath-\iota)=y\}\\
&= \sum_{\iota=1}^{\jmath-1}\mathbb{P}_x\{\sigma_{2l}=\iota\}\left(\frac{p}{q}\right)^{y-a}\mathbb{P}_a\{S(\jmath-\iota)=2a-y\}\\
&= \left(\frac{p}{q}\right)^{y-a}\mathbb{P}_x\{S(\jmath)=2a-y,\sigma_{2l}<\jmath\}\\
&= \left(\frac{p}{q}\right)^{y-a}\mathbb{P}_x\{S(\jmath)=2a-y,\sigma_{2l-1}<\jmath\}. \qquad \text{(B.4)}
\end{aligned}
$$

In the last step, we used the fact than, when $x,y\in(a,b)$, $2a-y<a$ and then the condition $S(\jmath)=2a-y,\sigma_{2l-1}<\jmath$ is enough to assure $\sigma(2l)<\jmath$. Similarly, reflection with respect to level b yields

$$
\begin{aligned}
&\mathbb{P}_x\{S(\jmath)=2a-y,\sigma_{2l-1}<\jmath\}\\
&\qquad= \left(\frac{p}{q}\right)^{2a-b-y}\mathbb{P}_x\{S(\jmath)=y+2(b-a),\sigma_{2l-2}<\jmath\}. \qquad \text{(B.5)}
\end{aligned}
$$

So, the following recursive relationship entails from (B.4) and (B.5):

$$
\mathbb{P}_x\{S(\jmath)=y,\sigma_{2l}<\jmath\} = \left(\frac{p}{q}\right)^{a-b}\mathbb{P}_x\{S(\jmath)=y+2(b-a),\sigma_{2l-2}<\jmath\}. \qquad \text{(B.6)}
$$

By iterating (B.6) with respect to index l, we plainly obtain

$$
\mathbb{P}_x\{S(\jmath)=y,\sigma_{2l}<\jmath\} = \left(\frac{p}{q}\right)^{l(a-b)}\mathbb{P}_x\{S(\jmath)=y+2l(b-a),\tau_a<\jmath\}. \qquad \text{(B.7)}
$$

A last application of the principle of reflection to (B.7) supplies

$$
\mathbb{P}_x\{S(\jmath)=y,\sigma_{2l}<\jmath\} = \left(\frac{p}{q}\right)^{y-a+l(b-a)}\mathbb{P}_x\{S(\jmath)=2a-2l(b-a)-y\}. \qquad \text{(B.8)}
$$

Analogously, concerning the second term of the sum (B.3), we successively have

$$
\begin{aligned}
&\mathbb{P}_x\{S(\jmath)=y,\sigma_{2l+1}<\jmath\}\\
&\qquad= \left(\frac{p}{q}\right)^{l(b-a)}\mathbb{P}_x\{S(\jmath)=y-2l(b-a),\sigma_1<\jmath\}\\
&\qquad= \left(\frac{p}{q}\right)^{y-b-l(b-a)}\mathbb{P}_x\{S(\jmath)=2b+2l(b-a)-y,\tau_a<\jmath\}\\
&\qquad= \left(\frac{p}{q}\right)^{(l+1)(b-a)}\mathbb{P}_x\{S(\jmath)=y-2(l+1)(b-a)\}. \qquad \text{(B.9)}
\end{aligned}
$$

We obtain in a quite similar manner the two last terms of (B.3):

$$\begin{aligned}
\mathbb{P}_x\{S(\jmath)=y,\varsigma_{2l}<\jmath\} \\
&= \left(\frac{p}{q}\right)^{y-b-l(b-a)} \mathbb{P}_x\{S(\jmath)=2b+2l(b-a)-y\} \\
&= \left(\frac{p}{q}\right)^{y-a-(l+1)(b-a)} \mathbb{P}_x\{S(\jmath)=2a+2(l+1)(b-a)-y\}, \\
\mathbb{P}_x\{S(\jmath)=y,\varsigma_{2l+1}<\jmath\} \\
&= \left(\frac{p}{q}\right)^{-(l+1)(b-a)} \mathbb{P}_x\{S(\jmath)=y+2(l+1)(b-a)\}.
\end{aligned} \tag{B.10}$$

Finally, by summing (B.8), (B.9) and (B.10), we deduce from (B.1) and (B.3):

$$\begin{aligned}
\mathbb{P}_x\{S(\jmath)=y,\tau_{a,b}<\jmath\} \\
&= \sum_{l=0}^{\infty}\left[\left(\frac{p}{q}\right)^{y-a+l(b-a)} p^{(\jmath)}_{x,2a-2l(b-a)-y} - \left(\frac{p}{q}\right)^{(l+1)(b-a)} p^{(\jmath)}_{x,y-2(l+1)(b-a)}\right. \\
&\quad \left. + \left(\frac{p}{q}\right)^{y-a-(l+1)(b-a)} p^{(\jmath)}_{x,2a+2(l+1)(b-a)-y} - \left(\frac{p}{q}\right)^{-(l+1)(b-a)} p^{(\jmath)}_{x,y+2(l+1)(b-a)}\right] \\
&= \sum_{l=-\infty}^{\infty}\left(\frac{p}{q}\right)^{l(b-a)}\left[\left(\frac{p}{q}\right)^{y-a} p^{(\jmath)}_{x,2a-2l(b-a)-y} - p^{(\jmath)}_{x,y-2l(b-a)}\right] + \mathbb{P}_x\{S(\jmath)=y\}
\end{aligned}$$

from which (1.15) ensues.

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